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GIMRADA Research Note No. 3

DETERMI ATION OF THE GEOMETRICAL QUALITY OF COMPARATORS FOR IMAGE COORDINATE MEASUREMENTS

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1 August 1962

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GEODESY, INTELLIGENCE AND MAPPING
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Task 8T35-12-001-02

Distributed by

The Director
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency

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CONTENTS

SUMMARY

This report covers the development and practical application of test procedures to determine the geometrical quality of comparators for image coordinate measurement. The procedures are founded upon grid coordinate measurements under operational conditions. First the basic principles for the determination of the accuracy of the measurements have been treated under different assumptions concerning the number and the positions of the test points. The principles of the method of least squares have been applied throughout, for the determination of regular (systematic) errors of the measured data as well as for the estimation of a statistical value of the irregular errors and for the error propagation in functions of the basic observations.

The derivations have been made for grids, the given coordinates of which can be regarded to be errorless and for grids where certain regular errors are assumed to be present in the given coordinates. In the latter case no absolute scale can be determined and the basic geometrical quality is in principle the precision.

In a series of practical applications the theoretical derivations have been used for testing a number of comparators of different types. Also some determinations of absolute scales have been performed. The lowest standard error of unit weight found in a comparator is of the order of magnitude 1 micron. Normal distribution tests of the residuals have been performed throughout.

From studies of the residuals further regular errors may be detected and determined.

DETERMINATION OF THE GEOMETRICAL QUALITY OF COMPARATORS

FOR IMAGE COORDINATE MEASUREMENTS

I. INTRODUCTION

Comparators for image coordinate measurements are of basic importance for all photogrammetric activity. Such instruments are used, for instance, in basic measurements for camera calibration and for tests of the basic geometrical quality of photographs for measuring purposes. For the calibration and test of various types of photogrammetric plotting instruments, glass grids are of basic importance. The coordinates of the intersections of the grid lines have to be determined by measurements in comparators. Further, analytical photogrammetry is founded upon measurements of image coordinates in comparators. Stereocomparators can be regarded as a set of two, sometimes three, individual comparators where the settings can be made stereoscopically and where parallaxes (image coordinate differences) sometimes are directly measured, in addition to the image coordinates of the individual photographs. For these and other purposes, the quality of the measurements in the comparator is of basic importance. It is necessary to determine and express this quality in a well-defined and easily understood manner. In connection with the delivery of a comparator, it is also necessary to check the real quality of the instrument in order to find out if the specifications which were agreed upon in the order contract are fulfilled. During practical work, it is also necessary to check the quality of the comparator to decide if adjustments have to be undertaken and to find out if the instrument works with the necessary reliability.

First, it is of interest to define the geometrical quality of a measuring instrument. The accuracy is to be distinguished from the precision. Accuracy has to be determined from a comparison between measured data and the corresponding given or known values of considerably higher reliability than the measured data. From the comparison, systematic or regular errors are first to be determined; i. e., such errors which follow a certain well-defined law concerning their direction and magnitude. The residual accidental or irregular errors, which in principle, do not follow any definable law but appear according to chance only, are assumed to be distributed according to the normal frequency function concerning the relation between the magnitude and the frequency. Such errors, consequently, define the ultimate accuracy and have to be estimated in a clear and well-defined manner. The most effective estimation is performed with the aid of the method of least squares as the standard error of unit weight. The accuracy of an instrument should, therefore, be expressed in terms of the

regular errors and the standard error of unit weight. It is also of value to show the distribution of the residual individual errors or discrepancies and to test the normal distribution to be expected.

In this connection, it is also appropriate to define the concept precision which frequently is used in literature. Here, precision refers to the comparison between replicated or repeated measurements or observations of unknown quantities. The precision is usually determined from the deviations between the average of a set of measurements and the measurements themselves and is expressed as standard deviation of one measurement and of the average. Since a comparator is to be used primarily for the measurements of image coordinates in a plane (x' and y'), the precision of the instrument must refer to the repeatability in settings of the coordinates of one or more points while the accuracy refers to the ability with which the instrument can determine the absolute coordinates of a grid of high and known quality (accuracy). The grid coordinates must, however, have been determined with another instrument and cannot, therefore, always be regarded to be completely errorless themselves. This will, of course, complicate the determination of the accuracy of the comparator to a certain extent as will be treated below. With respect to these circumstances, the determination of the geometrical quality will be performed under two different conditions concerning the quality of the given grid coordinates.

II. GENERAL PRINCIPLES OF THE DETERMINATION OF THE ACCURACY, ASSUMING THE GRID COORDINATES TO BE ERRORLESS

The comparator is assumed to be of high quality concerning the mechanical and optical parts. This means that the systematic errors are comparatively small and that the irregular or random errors are still smaller and at least approximately normally distributed. This means, for instance, that gross "jumps" in the rollers, rods, or ball bearings are not to be expected and, further, that the optical system is well manufactured and adjusted. The scales and reading devices must also be of uniformly high quality, without gross irregularities.

The regular errors of the instrument that are to be expected are scale differences in the two coordinate directions; absolute scale errors; and lack of orthogonality between the axes. If a grid is inserted into the plate holder of the comparator and the coordinates of the grid are measured, the lacking adjustment of the position of the grid can also be regarded as regular errors of the measured coordinates. The position of the grid is defined by three elements--two translations and one rotation of the grid. For the treatment of the accuracy of the comparator, the following

parameters are, therefore, regarded to cause regular or systematic errors of the measured grid coordinates:

2 translations of the grid: dx_o and dy_o

1 rotation of the grid: $d\alpha$
1 additional rotation of one of the axes of the comparator: $d\beta$
2 scale errors of the comparator: dm_x dm_y.

These factors seem to be of primary interest but evidently more sources of regular errors can be assumed and taken into account. Next, the differential relation between the six mentioned sources of regular errors and the errors of the measured coordinates will be derived, see Fig. 1.

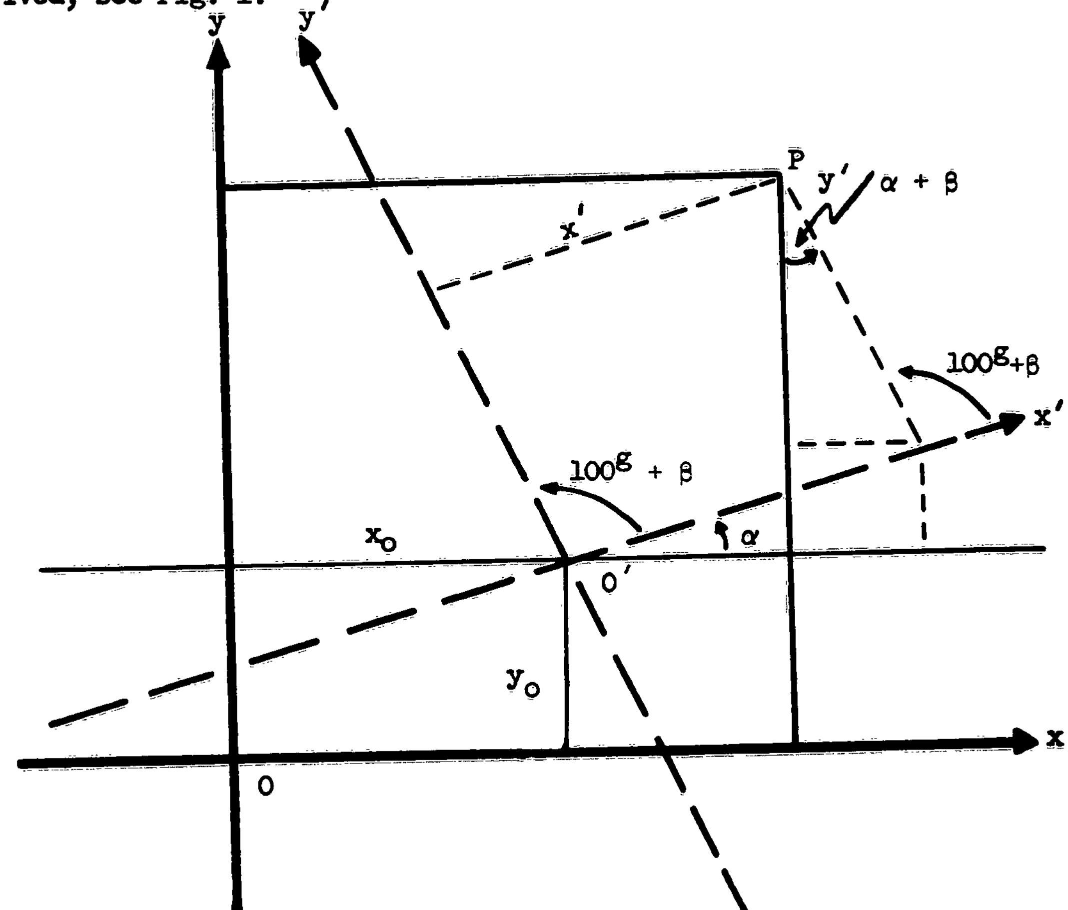


Figure 1. Coordinate transformation between the systems x', y' and x, y using the parameters x_o, y_o, the scale ratios

$m_x = x:x'$ and $m_y = y:y'$, the angles α and β.

Coordinate directions according to the Mann comparator.

From Fig. 1, the following relations are directly derived:

$$x = x_o + x'm_x\cos\alpha - y'm_y\sin(\alpha + \beta) \tag{1}$$

$$y = y_o + x'm_x\sin\alpha + y'm_y\cos(\alpha + \beta) \tag{2}$$

Through differentiation, we find:

$$dx = dx_o + x'\cos\alpha dm_x - x'm_x\sin\alpha d\alpha - y'\sin(\alpha+\beta)dm_y - - y'm_y\cos(\alpha+\beta)(d\alpha+d\beta) \tag{3}$$

$$dy = dy_o + x'\sin\alpha dm_x + x'm_x\cos\alpha d\alpha + y'\cos(\alpha+\beta)dm_y - -y'm_y\sin(\alpha+\beta)(d\alpha+d\beta) \tag{4}$$

For the approximative values: $m_x \approx 1$, $m_y \approx 1$, $\alpha \approx 0$, and $\beta \approx 0$, the differential formulas become:

$$dx = dx_o + xdm_x - y(d\alpha + d\beta) \tag{5}$$

$$dy = dy_o + ydm_y + xd\alpha \tag{6}$$

These formulas contain differentials of the six parameters, three of which (dx_o, dy_o, and $d\alpha$) affect the adjustment of the grid in the comparator while the dm_x, dm_y, and $d\beta$ in the case

of correct grid coordinates affect the comparator. If the comparator itself can be regarded as free from errors, these three parameters can then be referred to the grid. As will be discussed and treated below, these parameters can be referred to both the instrument and the grid and the problem of distinguishing between them will be treated in detail.

If an errorless grid is measured in a comparator, at least three points must be used for the determination of the six parameters since each point consists of two coordinates. Certain conditions exist for the solution of the six equations of the type (5) and (6) which means that there are conditions for the locations of the points. For the solution of the six equations, the discrepancies dx and dy of the expressions (5) and (6) are defined as errors; i. e.:

$$dx = x_{measured} - x_{given} \tag{7}$$

$$dy = y_{measured} - y_{given} \tag{8}$$

If the differentials of the expressions (5) and (6) are regarded as errors of the parameters, the expressions are denoted error equations. The corrections are reversed in sign and can be determined directly from the solved errors by changing the signs. This change can already be included in the expressions (5) and (6), and they are then denoted correction equations. For clearness, it is suitable to change the signs on the right side of the expressions (5) and (6).

If measurements have been made in more than the three necessary points, there are redundant observations available and no unique solution can be obtained from different sets of three points. In such a case, the method of least squares is the most convenient way to treat the problem of determining unique values of the errors or corrections of the parameters. The observed discrepancies dx and dy are corrected with small quantities v_x and v_y, and the working correction equations are obtained from (5) and (6) as follows:

$$v_x = -dx_o - xdm_x + y(d\alpha + d\beta) - dx \tag{9}$$

$$v_y = -dy_o - ydm_y - xd\alpha - dy \tag{10}$$

dx and dy are defined according to expressions (7) and (8) above. From the working correction equations, the normal equations are derived and solved. The corrections to the parameters are then obtained as direct functions of the measured discrepancies and under the conditions that the sum of the squares $[v_x v_x] + [v_y v_y]$ becomes a minimum. From the solutions of the normal equations, also the expression for this square sum and the weight and correlation numbers are obtained. Residuals after the adjustment can be computed from the original working correction equations (9) and (10) after substituting the solved corrections. All functions of the adjusted quantities can be treated concerning corrections, and the accuracy of these corrections can also be determined from the laws of error propagation. The complete theory of errors for the results of the actual measurements can, in other words, be treated and the resultant accuracy can be expressed in terms of standard errors and as functions of the basic standard error of unit weight, which can be determined from the sum of the squares $[vv] = [v_x v_x] + [v_y v_y]$ according to the formula

$$s_o = \sqrt{\frac{[vv]}{2n-6}} \tag{11}$$

where n is the number of points used in the adjustment procedure. The standard error of the standard error of unit weight is found from the expression $s_{s_o} = \dfrac{s_o}{\sqrt{2(2n-6)}}$ (12)

The confidence limits for the standard error of unit weight and for any function can be determined from the procedures shown in mathematical statistics.

Practical applications of these principles will be demonstrated below.

1. Measurements in Three Points. In this case, no adjustment is required since no redundant measurements are available. In particular for a demonstration of the error propagation procedure, a complete treatment of this case will be made. The three points are assumed to be located as shown in Fig. 2.

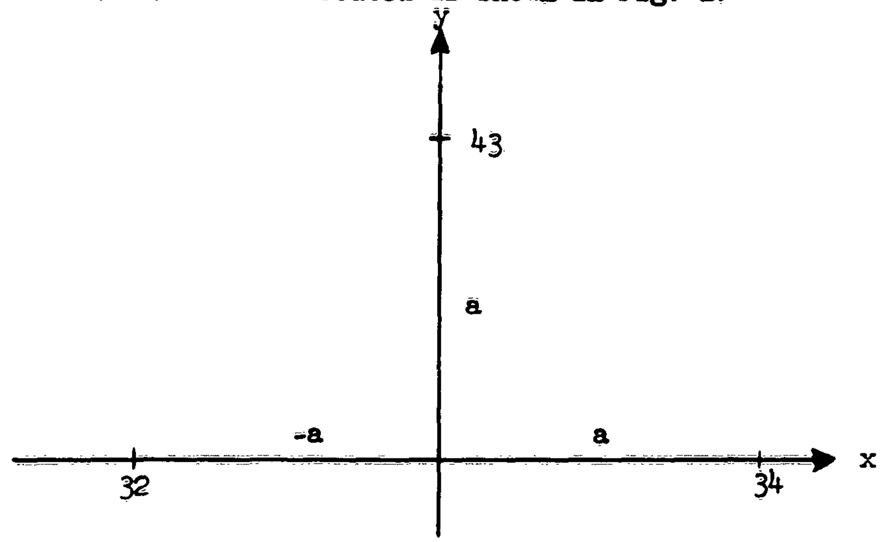


Figure 2. The assumed locations and notations of three points.

The coordinates of the three points are:

Point	x	y
34	a	0
32	-a	0
43	0	a

(13)

The correction equations are from (5) and (6):

$$dx = -dx_o - xdm_x + y(d\alpha + d\beta) \tag{14}$$

$$dy = -dy_o - ydm_y - xd\alpha \tag{15}$$

Applied to the three points (13) and the corresponding discrepancies, the expressions (14) and (15) become:

$$dx_{34} = -dx_o - adm_x$$
$$dy_{34} = -dy_o - ad\alpha$$

$$dx_{32} = -dx_o + adm_x$$
$$dy_{32} = -dy_o + ad\alpha$$

$$dx_{43} = -dx_o + a(d\alpha + d\beta)$$
$$dy_{43} = -dy_o - adm_y \tag{16}$$

The solution of this system is:

$$dx_o = -\frac{dx_{34}+dx_{32}}{2} \tag{17}$$

$$dy_o = -\frac{dy_{43}+dy_{32}}{2} \tag{18}$$

$$dm_x = \frac{dx_{32}-dx_{34}}{2a} \tag{19}$$

$$dm_y = \frac{dy_{34}+dy_{32}-2dy_{43}}{2a} \tag{20}$$

$$d\alpha = \frac{dy_{32}-dy_{34}}{2a} \tag{21}$$

$$d\beta = \frac{2dx_{43}-dx_{32}-dx_{34}+dy_{34}-dy_{32}}{2a} \tag{22}$$

According to their definitions, the weight and correlation numbers are:

$$Q_{x_ox_o} = Q_{y_oy_o} = \frac{1}{2} \qquad Q_{\beta\beta} = \frac{2}{a^2}$$

$$Q_{m_xm_x} = \frac{1}{2a^2} \qquad Q_{x_o\beta} = \frac{1}{2a}$$

$$Q_{m_ym_y} = \frac{3}{2a^2} \qquad Q_{y_om_y} = -\frac{1}{2a}$$

$$Q_{\alpha\alpha} = \frac{1}{2a^2} \qquad Q_{\alpha\beta} = -\frac{1}{2a^2} \tag{23}$$

In this case, no determination of the standard error of unit weight can be made since no redundant observations are available. For the error propagation, the basic standard error of unit weight, therefore, must be obtained from other experiments where redundant measurements are available.

From the expressions (14) and (15), corrections can be computed to arbitrary points in which measurements were made simultaneously with the measurements in the points 32, 34, and 43.

The accuracy of these corrections can be determined from an application of the general law of error propagation to the expressions (14) and (15). The weight numbers of the corrections are:

$$Q_{dxdx} = Q_{x_o x_o} + x^2 Q_{m_x m_x} + y^2 Q_{\alpha\alpha} + y^2 Q_{\beta\beta} - 2y Q_{x_o \beta} + 2y^2 Q_{\alpha\beta} \quad (24)$$

$$Q_{dydy} = Q_{y_o y_o} + y^2 Q_{m_y m_y} + x^2 Q_{\alpha\alpha} + 2y\, Q_{y_o m_y} \quad (25)$$

The weight number of a corrected coordinate is found after addition of the weight number of the measurement of the coordinate in question. This weight number is assumed to be 1; the weight number of a corrected coordinate can, therefore, after substitution of the weight and correlation numbers (23), be expressed as:

$$Q_{xx} = \frac{3}{2} + \frac{x^2}{2a^2} + \frac{3y^2}{2a^2} - \frac{y}{a} \quad (26)$$

$$Q_{yy} = \frac{3}{2} \div \frac{x^2}{2a^2} + \frac{3y^2}{2a^2} - \frac{y}{a} \quad (27)$$

The standard errors of the corrected coordinates are then found from:

$$s_x = s_o \sqrt{Q_{xx}} \quad (28)$$

$$s_y = s_o \sqrt{Q_{yy}} \quad (29)$$

s_o is the standard error of unit weight of the coordinate measurements. The distribution of the standard errors can be found from a graphical plotting of (28) and (29). For simplicity, the s_o is chosen = 1.

First, it is suitable to determine characteristic points as, for instance, possible maxima and minima. From differentiation of the expressions (26) and (27), we find:

$$\frac{dQ_{xx}}{dx} = \frac{x}{a^2}$$

$$\frac{dQ_{xx}}{dx} = \frac{3y}{a^2} - \frac{1}{a}$$

$$\frac{d^2Q_{xx}}{dx^2} = \frac{1}{a^2}$$

$$\frac{d^2Q_{yy}}{dy^2} = \frac{3}{a^2}$$

Since both the second derivatives are positive, there is a minimum for the point:

$$x = 0,\ y = \frac{a}{3}$$

Similarly, a minimum point is found for y, having the same location. The standard error at this point is 1.15. In Fig. 3, a graphical plotting of the expressions (28) and (29) is shown. From such a plot, the root mean square value of the errors within a certain area can be determined by estimation. It is also possible to make such a determination of the root mean square values of the theoretical errors by mathematical means. If, for instance, the root mean square value of the standard errors of x or y is to be determined within the area

$$-a \leq x \leq +a$$

$$-a \leq y \leq +a$$

the following expressions are used:

$$M_{s_x}^2 = M_{s_y}^2 = \frac{1}{4a^2}\, s_o^2 \int_{x=-a}^{x=+a} \int_{y=-a}^{y=+a} \left(\frac{3}{2} + \frac{x^2}{2a^2} + \frac{3y^2}{2a^2} - \frac{y}{a} \right) dxdy \qquad (30)$$

After integration, we find the value $M_{s_x}^2 = M_{s_y}^2 = 1.67$

Hence

$$M_{s_x} = M_{s_y} = 1.3\ s_o \qquad (31)$$

In a practical experiment, this value can be compared with a root mean square value of the computed discrepancies. The agreement should not be expected to be exact but should be judged with respect

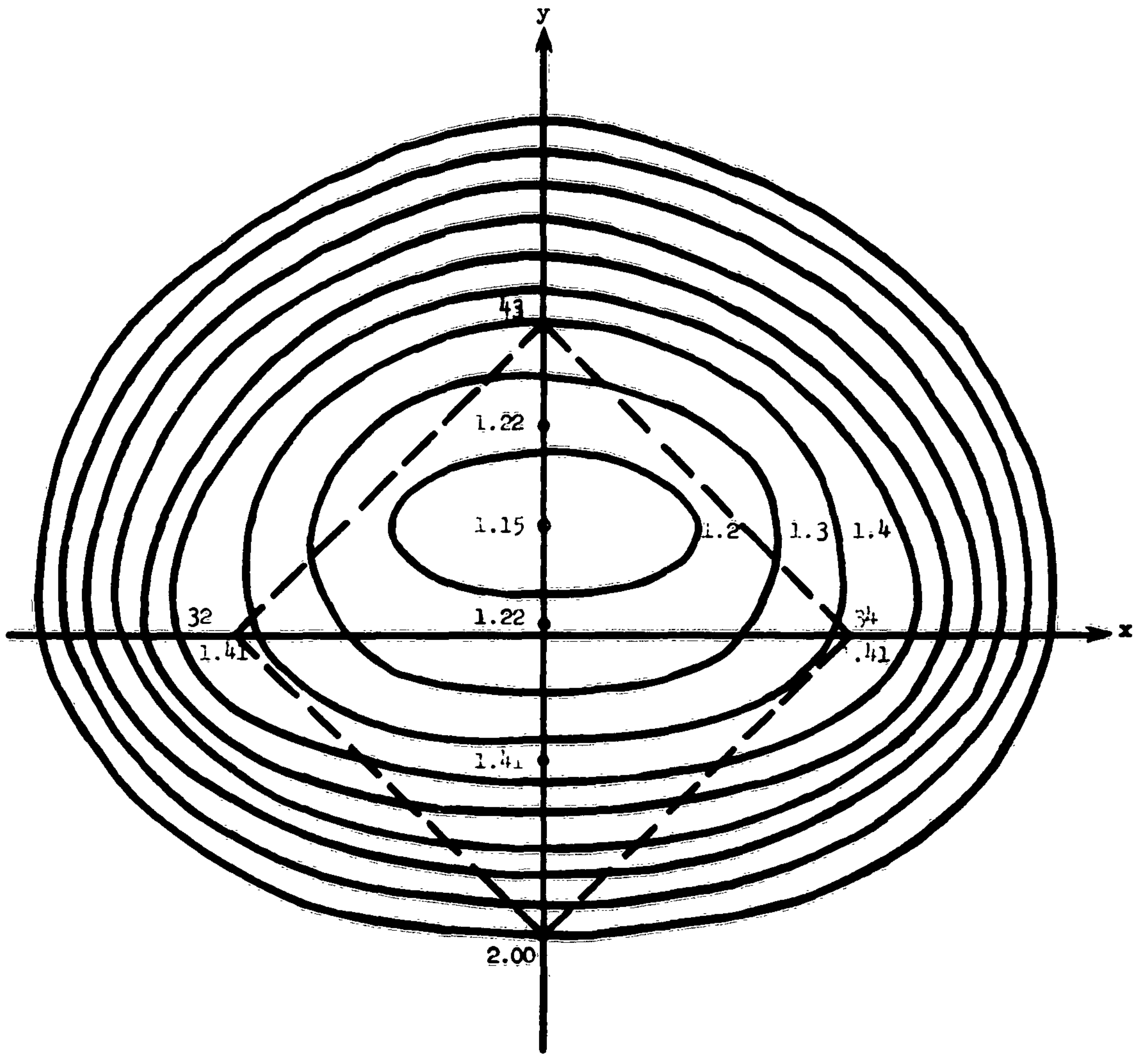


Figure 3. Distribution of the standard errors of corrected coordinates; corrections from measurements in three points; 32, 34, and 43; $s_o = 1$.

to the confidence limits of the theoretically determined value. Examples of the determination of such confidence limits will be given below.

2. Measurements in Four Points and Adjustment. The basic working correction equations are the expressions (9) and (10) above. We assume the points to be located as shown in Figure 4.

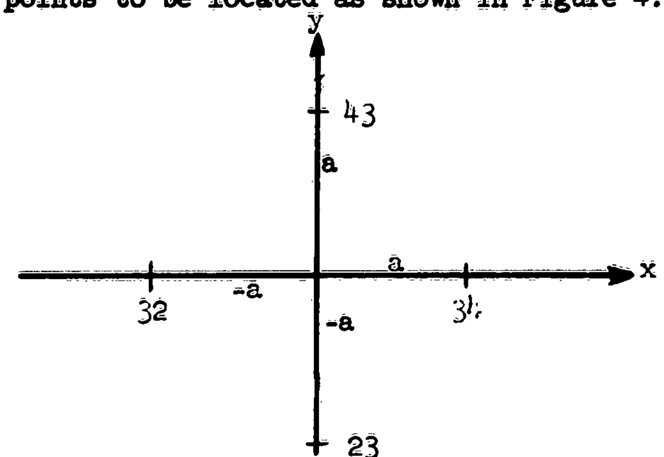


Figure 4. Locations and notations of the four points for coordinate measurements.

The working correction equations are applied to each of the points, and the coefficients of the differentials are tabulated as shown in Table I.

Table I. Working Correction Equations

	Point	x	y	dx_o	dy_o	dm_x	dm_y	$d\alpha$	$d\beta$	dx	dy
v_x	34	+a	0	-1	0	-a	0	0	0	$-dx_{34}$	0
	32	-a	0	-1	0	+a	0	0	0	$-dx_{32}$	0
	43	0	+a	-1	0	0	0	+a	+a	$-dx_{43}$	0
	23	0	-a	-1	0	0	0	-a	-a	$-dx_{23}$	0
v_y	34	+a	0	0	-1	0	0	-a	0	0	$-dy_{34}$
	32	-a	0	0	-1	0	0	+a	0	0	$-dy_{32}$
	43	0	+a	0	-1	0	-a	0	0	0	$-dy_{43}$
	23	0	-a	0	-1	0	+a	0	0	0	$-dy_{23}$

The normal equations are formed in the usual way and become:

$$4dx_o + [dx] = 0$$

$$4dy_o + [dy] = 0$$

$$2a^2dm_x + a(dx_{34} - dx_{32}) = 0$$

$$2a^2dm_y + a(dy_{43} - dy_{23}) = 0$$

$$4a^2d\alpha + 2a^2d\beta + a(dx_{23} - dx_{43} + dy_{34} - dy_{32}) = 0$$

$$2a^2d\alpha + 2a^2d\beta + a(dx_{23} - dx_{43}) = 0 \qquad (32)$$

The solutions of these equations are:

$$dx_o = -\frac{[dx]}{4}$$

$$dy_o = -\frac{[dy]}{4}$$

$$dm_x = \frac{dx_{32} - dx_{34}}{2a}$$

$$dm_y = \frac{dy_{23} - dy_{43}}{2a}$$

$$d\alpha = \frac{dy_{32} - dy_{34}}{2a}$$

$$d\beta = \frac{dx_{43} - dx_{23} + dy_{34} - dy_{32}}{2a} \qquad (33)$$

The weight and correlation numbers are:

$$Q_{x_o x_o} = Q_{y_o y_o} = \frac{1}{4} \qquad Q_{\alpha\alpha} = \frac{1}{2a^2}$$

$$Q_{m_x m_x} = Q_{m_y m_y} = \frac{1}{2a^2} \qquad Q_{\beta\beta} = \frac{1}{a^2} \qquad Q_{\alpha\beta} = -\frac{1}{2a^2} \qquad (34)$$

Further:

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{4} - \frac{(dx_{32} - dx_{34})^2 + (dy_{23} - dy_{43})^2}{2} -$$

$$- \frac{(dy_{34} - dy_{32})^2 + (dy_{43} - dy_{23})^2}{2} \qquad (35)$$

Hence

$$s_o = \sqrt{\frac{[vv]}{2}} \qquad (36)$$

The standard error of the standard error is, according to (12),

$$s_{s_o} = 0.50\, s_o \qquad (37)$$

The reliability of the standard error of unit weight is, consequently, rather low due to the low number of redundant observations (8 - 6 = 2). The confidence limits of s_o are:

for the five per cent level	$1.0s_o < s_o < 12.6s_o$
for the one per cent level	$0.9s_o < s_o < 28.3s_o$ (38)

The confidence limits of individual residuals after the adjustment are:

for the five per cent level	$\pm 4.3s$
for the one per cent level	$\pm 9.9s$ (39)

Consequently, rather large deviations of the standard error of unit weight from the usual (true) value must be accepted and also rather large residuals after corrections.

From the working correction equations (9) and (10), the standard errors of the residuals can be determined. After application of the general law of error propagation and using the weight and correlation numbers (34), the weight numbers of corrected coordinates are found as follows:

$$Q_{xx} = \frac{5}{4} + \frac{x^2 + y^2}{2a^2} = Q_{yy} \qquad (40)$$

Hence:

$$s_x = s_y = s_o \sqrt{\frac{5}{4} + \frac{x^2 + y^2}{2a^2}} \qquad (41)$$

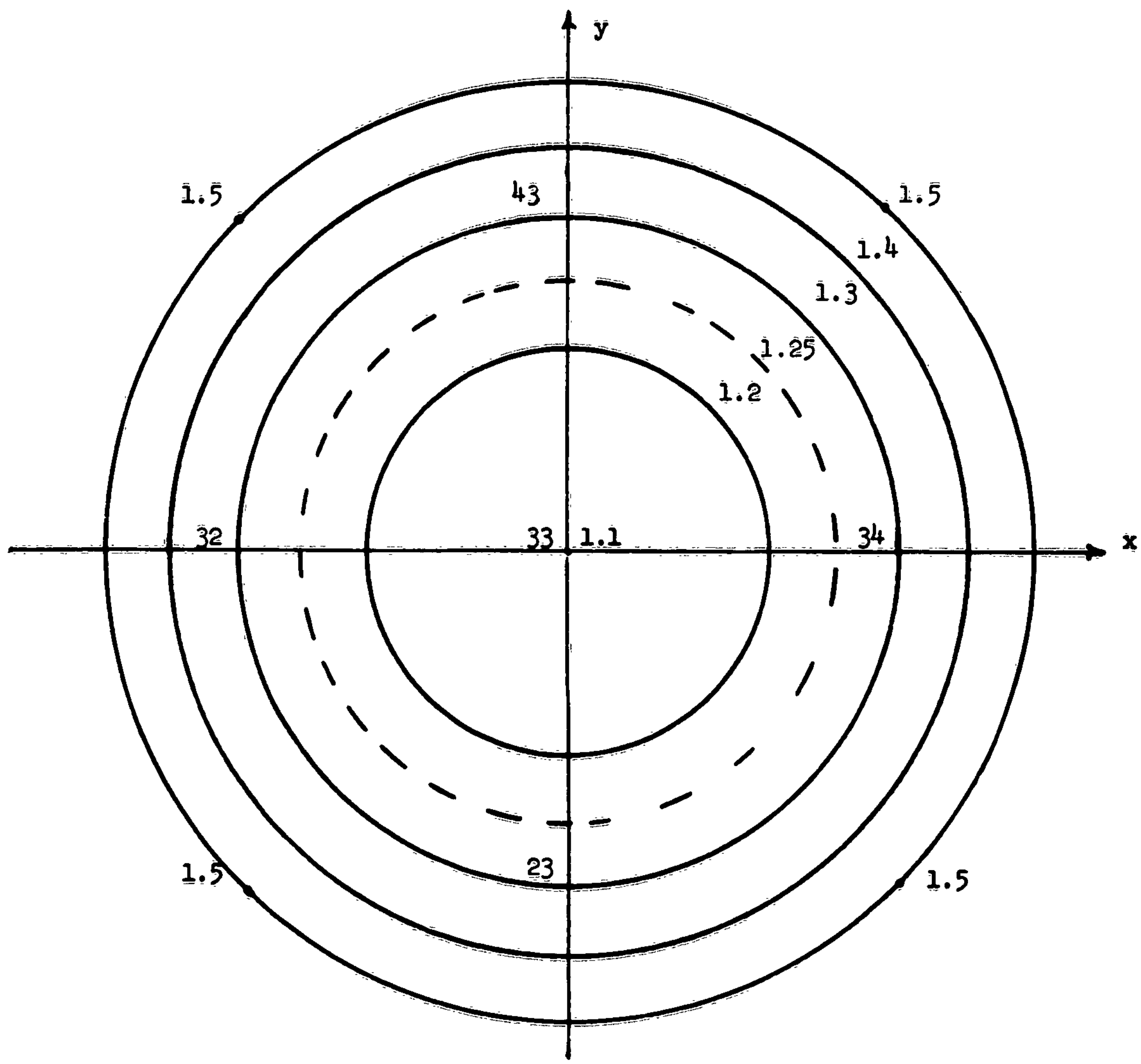


Figure 5. Distribution of standard errors of corrected coordinates; corrections from four points; $s_o = 1$.

These expressions are symmetrical around the coordinate origin, which is to be expected due to the symmetrical positions of the points. A graphical plotting of the expression (41) is shown in Fig. 5. The root mean square value of the standard errors within a certain area can be determined from Fig. 5 or from the numerical procedure. For the area 2a x 2a, the root mean square value is found as follows:

$$M_{sx}^2 = \frac{s_o^2}{4a^2} \int_{x=-a}^{x=+a} \int_{y=-a}^{y=+a} \left(\frac{5}{4} + \frac{x^2 + y^2}{2a^2}\right) dxdy \qquad (42)$$

After the integration, we find:

$$M_{sx} = M_{sy} = 1.25 s_o \qquad (43)$$

The computations of the adjustments are considerably facilitated if suitable forms are used(See forms 1 and 2, Appendix).

3. Measurements in Nine Points and Adjustment. We assume 25 of the grid points to be located and denoted as shown in Fig. 6.

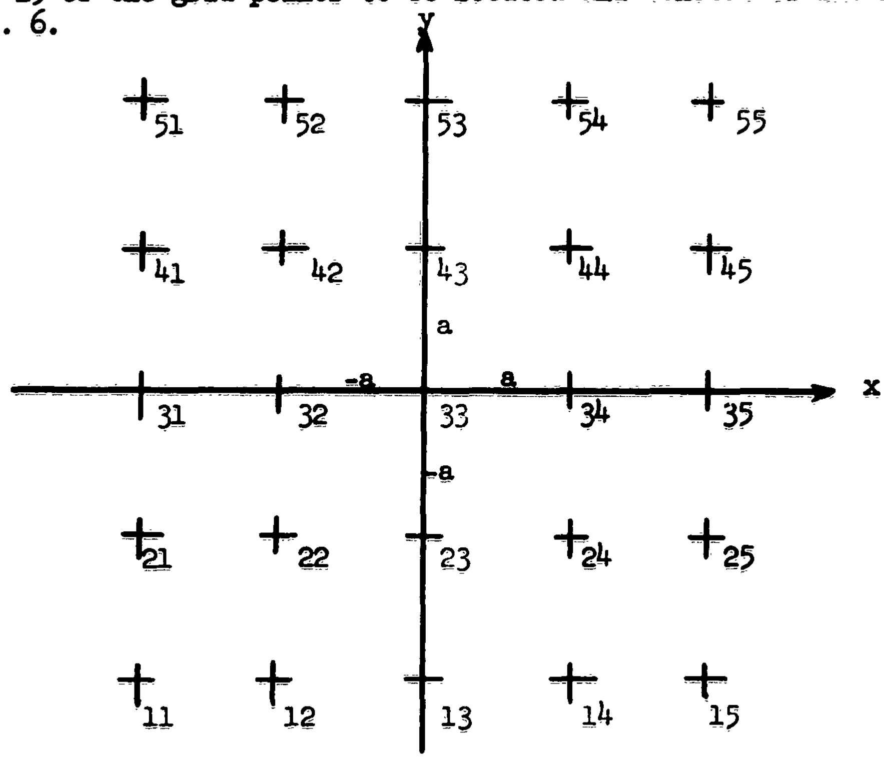


Figure 6. Locations and notations of 25 regular grid points.

For the treatment of a set of nine points, we use these: 11, 13, 15, 31, 33, 35, 51, 53, and 55. The procedure for the adjustment is similar to the procedure for four points shown in section II2. The working correction equations (9) and (10) are applied to each of the nine points. The results are shown in Table II.

Table II. Working Correction Equations for Nine Points

	Point	x	y	dx_o	dy_o	dm_x	dm_y	$d\alpha$	$d\beta$	dx	dy
v_x	11	-2a	-2a	-1	-	2a	-	-2a	-2a	$-dx_{11}$	-
	13	0	-2a	-1	-	-	-	-2a	-2a	$-dx_{13}$	-
	15	+2a	-2a	-1	-	-2a	-	-2a	-2a	$-dx_{15}$	-
	31	-2a	0	-1	-	2a	-	-	-	$-dx_{31}$	-
	33	0	0	-1	-	-	-	-	-	$-dx_{33}$	-
	35	+2a	0	-1	-	-2a	-	-	-	$-dx_{35}$	-
	51	-2a	+2a	-1	-	2a	-	2a	2a	$-dx_{51}$	-
	53	0	+2a	-1	-	-	-	2a	2a	$-dx_{53}$	-
	55	+2a	+2a	-1	-	-2a	-	2a	2a	$-dx_{55}$	-
v_y	11	-2a	-2a	-	-1	-	2a	2a	-	-	$-dy_{11}$
	13	0	-2a	-	-1	-	2a	-	-	-	$-dy_{13}$
	15	+2a	-2a	-	-1	-	2a	-2a	-	-	$-dy_{15}$
	31	-2a	0	-	-1	-	-	2a	-	-	$-dy_{31}$
	33	0	0	-	-1	-	-	-	-	-	$-dy_{33}$
	35	+2a	0	-	-1	-	-	-2a	-	-	$-dy_{35}$
	51	-2a	+2a	-	-1	-	-2a	2a	-	-	$-dy_{51}$
	53	0	+2a	-	-1	-	-2a	-	-	-	$-dy_{53}$
	55	+2a	+2a	-	-1	-	-2a	-2a	-	-	$-dy_{55}$

The normal equations become:

$$9dx_o + [dx] = 0$$

$$9dy_o + [dy] = 0$$

$$24a^2dm_x + 2aN91 = 0$$

$$24a^2dm_y + 2aN92 = 0$$

$$48a^2 d\alpha + 24a^2 d\beta + 2aN93 = 0$$

$$24a^2 d\alpha + 24a^2 d\beta + 2aN94 = 0 \qquad (44)$$

The symbols N91 through N94 are defined as follows:

$$N91 = -dx_{11} + dx_{15} - dx_{31} + dx_{35} - dx_{51} + dx_{55} \qquad (45)$$

$$N92 = -dy_{11} - dy_{13} - dy_{15} + dy_{51} + dy_{53} + dy_{55} \qquad (46)$$

$$N93 = dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55} - dy_{11} + dy_{15}$$
$$- dy_{31} + dy_{35} - dy_{51} + dy_{55} \qquad (47)$$

$$N94 = dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55} \qquad (48)$$

The solutions of the normal equations are:

$$dx_o = -\frac{[dx]}{9}$$

$$dy_o = -\frac{[dy]}{9}$$

$$dm_x = -\frac{N91}{12a}$$

$$dm_y = -\frac{N92}{12a}$$

$$d\alpha = \frac{N94 - N93}{12a}$$

$$d\beta = \frac{-2N94 + N93}{12a} \qquad (49)$$

The weight and correlation numbers are:

$$Q_{x_o x_o} = Q_{y_o y_o} = \frac{1}{9}$$

$$Q_{m_x m_x} = Q_{m_y m_y} = Q_{\alpha\alpha} = \frac{1}{24a^2}$$

$$Q_{\beta\beta} = \frac{1}{12a^2} \qquad Q_{\alpha\beta} = -\frac{1}{24a^2} \qquad (50)$$

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{9} - \frac{N91^2 + N92^2 + (N94-N93)^2 + N94^2}{6} \qquad (51)$$

$$s_o = \sqrt{\frac{[vv]}{12}} \tag{52}$$

$$s_{s_o} = 0.21 s_o \tag{53}$$

The confidence limits are for the standard error of unit weight:

for the five per cent level $\quad 0.9s_o - 2.0s_o$

for the one per cent level $\quad 0.8s_o - 3.2s_o \qquad (54)$

The confidence limits are for individual residuals after the adjustment and correction:

for the five per cent level $\quad \pm\ 2.2s$

for the one per cent level $\quad \pm\ 3.1s \qquad (55)$

The weight numbers of corrected coordinates are found from the application of the general law of error propagation to the working correction equations (9) and (10), using the weight and correlation numbers (50). After computations, we find:

$$Q_{xx} = Q_{yy} = \frac{10}{9} + \frac{x^2 + y^2}{24a^2} \tag{56}$$

The corresponding standard errors are

$$s_x = s_y = s_o \sqrt{\frac{10}{9} + \frac{x^2 + y^2}{24a^2}} \tag{57}$$

This expression is graphically shown in Fig. 7 for $s_o = 1$.

The root mean square value of the standard errors within the area $4a + 4a$ is found from the expression

$$M_{s_x}^2 = M_{s_y}^2 = \frac{s_o^2}{16a^2} \int_{x=-2a}^{x=+2a} \int_{y=-2a}^{y=+2a} \left(\frac{10}{9} + \frac{x^2 + y^2}{24a^2}\right) dxdy \tag{58}$$

After integration, we find

$$M_{s_x} = M_{s_y} = 1.1 s_o \tag{59}$$

For the practical computations, see forms 1 and 3 (Appendix).

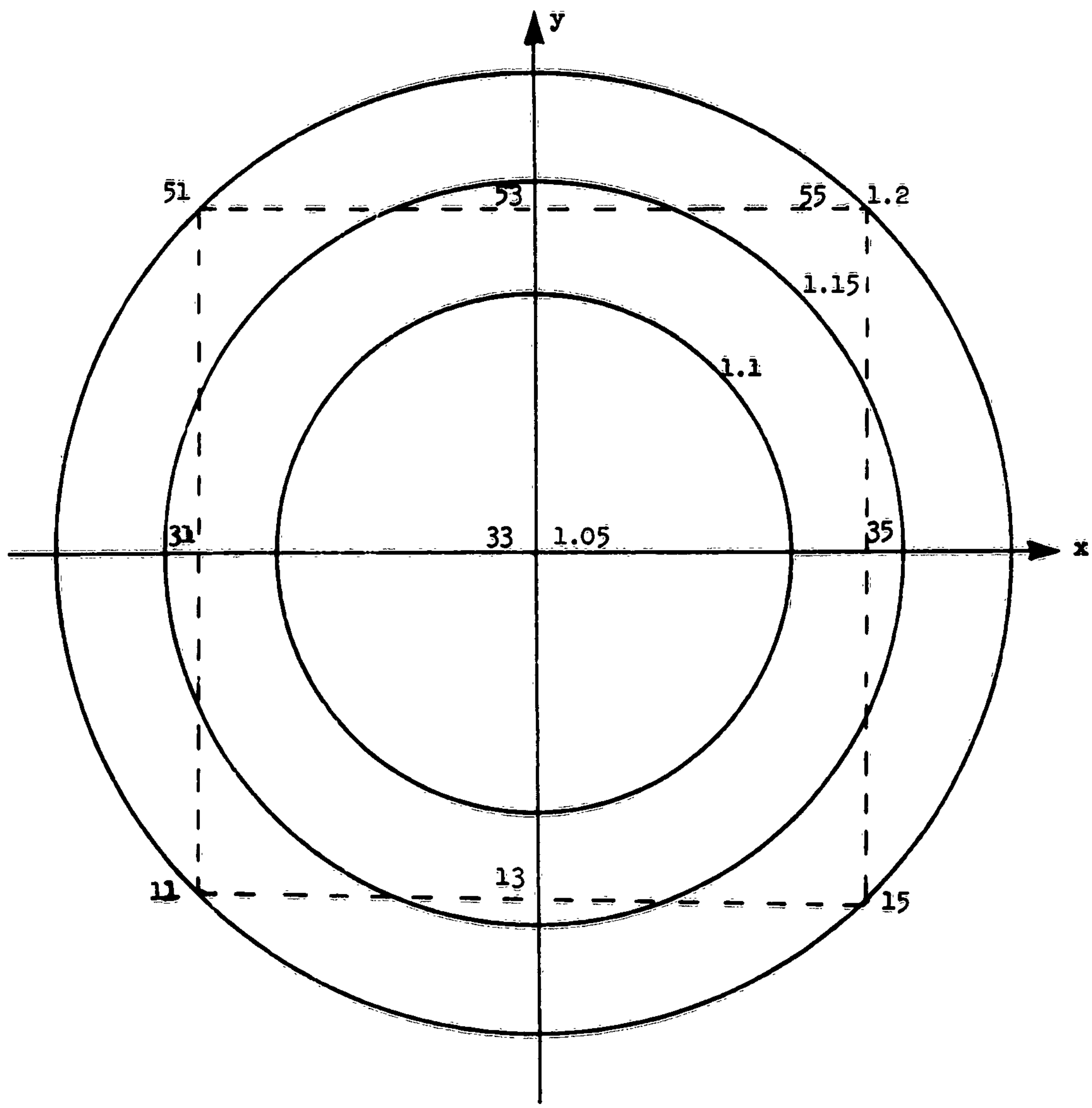


Figure 7. Distribution of standard errors of corrected coordinates; corrections determined from nine points; $s_o = 1$.

The main purpose of the following determination is to find the standard errors of the residuals in the actual points for a comparison with the actual residual. We use the working correction equations and substitute in these the expressions for the corrections from the adjustment and the coordinate discrepancies before the adjustment. After rather comprehensive computations (which are not given in detail here), the following expressions are found:

$$V_{x11} = \frac{1}{18}\left(-10dx_{11} + 5dx_{13} + 2dx_{15} + 5dx_{31} + 2dx_{33} - dx_{35} + 2dx_{51} - dx_{53} - 4dx_{55}\right) \tag{60}$$

$$V_{x13} = \frac{1}{18}\left(5dx_{11} - 13dx_{13} + 5dx_{15} + 2dx_{31} + 2dx_{33} + 2dx_{35} - dx_{51} - dx_{53} - dx_{55}\right) \tag{61}$$

$$V_{x15} = \frac{1}{18}\left(2dx_{11} + 5dx_{13} - 10dx_{15} - dx_{31} + 2dx_{33} + 5dx_{35} - 4dx_{51} - dx_{53} + 2dx_{55}\right) \tag{62}$$

$$V_{x_{31}} = \frac{1}{18}\left(5dx_{11} + 2dx_{13} - dx_{15} - 13dx_{31} + 2dx_{33} - dx_{35} + 5dx_{51} + 2dx_{53} - dx_{55}\right) \tag{63}$$

$$V_{x_{33}} = \frac{1}{9}\left(dx_{11} + dx_{13} + dx_{15} + dx_{31} - 8dx_{33} + dx_{35} + dx_{51} + dx_{53} + dx_{55}\right) \tag{64}$$

$$V_{x_{35}} = \frac{1}{18}\left(-dx_{11} + 2dx_{13} + 5dx_{15} - dx_{31} + 2dx_{33} - 13dx_{35} - dx_{51} + 2dx_{53} + 5dx_{55}\right) \tag{65}$$

$$V_{x_{51}} = \frac{1}{18}\left(2dx_{11} - dx_{13} - 4dx_{15} + 5dx_{31} + 2dx_{33} - dx_{35} - 10dx_{51} + 5dx_{53} + 2dx_{55}\right) \tag{66}$$

$$V_{x_{53}} = \frac{1}{18} (-dx_{11} - dx_{13} - dx_{15} + 2dx_{31} + 2dx_{33} + 2dx_{35} + 5dx_{51} -$$

$$- 13dx_{53} + 5dx_{55}) \tag{67}$$

$$V_{x_{55}} = \frac{1}{18} (-4dx_{11} - dx_{13} + 2dx_{15} - dx_{31} + 2dx_{33} + 5dx_{35} + 2dx_{51} +$$

$$+ 5dx_{53} - 10dx_{55}) \tag{68}$$

Next, the weight numbers of the residuals can be computed as the square sums of the coefficients of the measured data. From the expressions (60) through (68), we find the weight numbers and standard errors as shown in Fig. 8.

$^{51}Q \doteq \frac{5}{9}$ $s = 0.75\ s_o$	$^{53}Q \doteq \frac{13}{18}$ $s = 0.85\ s_o$	$^{55}Q \doteq \frac{5}{9}$ $s = 0.75\ s_o$
$^{31}Q \doteq \frac{13}{18}$ $s = 0.85\ s_o$	$^{33}Q \doteq \frac{8}{9}$ $s = 0.94\ s_o$	$^{35}Q \doteq \frac{13}{18}$ $s = 0.85\ s_o$
$^{11}Q \doteq \frac{5}{9}$ $s = 0.75\ s_o$	$^{13}Q \doteq \frac{13}{18}$ $s = 0.85\ s_o$	$^{15}Q \doteq \frac{5}{9}$ $s = 0.75\ s_o$

Figure 8. Weight numbers and standard errors of the residuals in the nine points after adjustment.

4. Measurements in 25 Points and Adjustment. More observations are desired in order to increase the reliability of the determination of the basic accuracy (the standard errors of unit weight) of the coordinate measurements in the comparator. The reliability of the standard error of unit weight can be expressed by the standard error of the standard error of unit weight which is

$$s_{s_o} = \frac{s_o}{\sqrt{2r}} \tag{69}$$

where r is the number of redundant measurements.

For measurements in four points, we have $r = 2$ and consequently $s_{s_{o4}} = 0.5s_o$.

For nine points $r = 12$ and $s_{s_{o9}} = 0.21s_o$.

For 25 points $r = 44$ and $s_{s_{o25}} = 0.11s_o$.

For further increase in the number of points, the decrease of the standard error of the standard error of unit weight becomes rather small and does not justify the increased amount of work and time consumption for the measurements and computations. Under normal circumstances, therefore, measurements in 25 points can be regarded as a maximum.

The procedure for the treatment of the measured data and for the discrepancies is very similar to what has been applied to nine points above and will not be repeated in detail here.

After the working correction equations are tabulated according to expressions (9) and (10) and the coordinates of Fig. 6 are used for the 25 points, the normal equations become as follows:

$$\begin{aligned}
&25dx_o + [dx] = 0\\
&25dy_o + [dy] = 0\\
&50a^2dm_x + aN251 = 0\\
&50a^2dm_y + aN252 = 0\\
&100a^2d\alpha + 50a^2d\beta + aN253 = 0\\
&50a^2d\alpha + 50a^2d\beta + aN254 = 0
\end{aligned} \tag{70}$$

The solution is:

$$dx_o = -\frac{[dx]}{25} \qquad Q_{x_o x_o} = Q_{y_o y_o} = \frac{1}{25}$$

$$dy_o = -\frac{[dy]}{25}$$

$$dm_x = -\frac{N251}{50a} \qquad Q_{m_x m_x} = Q_{m_y m_y} = Q_{\alpha\alpha} = \frac{1}{50a^2}$$

$$dm_y = -\frac{N252}{50a}$$

$$d\alpha = \frac{N254-N253}{50a} \qquad Q_{\alpha\beta} = -\frac{1}{50a^2}$$

$$d\beta = \frac{N253-2N254}{50a} \qquad Q_{\beta\beta} = \frac{1}{25a^2} \tag{71}$$

$$[vv] = [dxdx] + [dydy] - \frac{[dx]^2 + [dy]^2}{25} - \frac{N251^2 + N252^2 + (N253-N254)^2 +}{50}$$

$$+ \frac{N254^2}{50} \tag{72}$$

$$s_o = \sqrt{\frac{[vv]}{44}} \tag{73}$$

$$s_{s_o} = \frac{s_o}{\sqrt{88}} = 0.11 s_o \tag{74}$$

The symbols are defined as follows:

$$N251 = -2dx_{11} - dx_{12} + dx_{14} + 2dx_{15} - 2dx_{21} - dx_{22} + dx_{24} + 2dx_{25} -$$
$$- 2dx_{31} - dx_{32} + dx_{34} + 2dx_{35} - 2dx_{41} - dx_{42} + dx_{44} + 2dx_{45} -$$
$$-2dx_{51} - dx_{52} + dx_{54} + 2dx_{55} \tag{75}$$

$$N252 = -2dy_{11} - 2dy_{12} - 2dy_{13} - 2dy_{14} - 2dy_{15} - dy_{21} - dy_{22} - dy_{23} - dy_{24} - dy_{25} + dy_{41} + dy_{42} + dy_{43} + dy_{44} + dy_{45} + 2dy_{51} + 2dy_{52} + 2dy_{53} + 2dy_{54} + 2dy_{55} \tag{76}$$

$$N253 = 2dx_{11} + 2dx_{12} + 2dx_{13} + 2dx_{14} + 2dx_{15} + dx_{21} + dx_{22} + dx_{23} + dx_{24} + dx_{25} - dx_{41} - dx_{42} - dx_{43} - dx_{44} - dx_{45} - 2dx_{51} - 2dx_{52} - 2dx_{53} - 2dx_{54} - 2dx_{55} - 2dy_{11} - dy_{12} + dy_{14} + 2dy_{15} - 2dy_{21} - dy_{22} + dy_{24} + 2dy_{25} - 2dy_{31} - dy_{32} + dy_{34} + 2dy_{35} - 2dy_{41} - dy_{42} + dy_{44} + 2dy_{45} - 2dy_{51} - dy_{52} + dy_{54} + 2dy_{55} \tag{77}$$

$$N254 = 2dx_{11} + 2dx_{12} + 2dx_{13} + 2dx_{14} + 2dx_{15} + dx_{21} + dx_{22} + dx_{23} + dx_{24} + dx_{25} - dx_{41} - dx_{42} - dx_{43} - dx_{44} - dx_{45} - 2dx_{51} - 2dx_{52} - 2dx_{53} - 2dx_{54} - 2dx_{55} \tag{78}$$

$dx = x_{measured} - x_{given}$; $dy = y_{measured} - y_{given}$.

The confidence limits of s_o are

for five per cent level $0.9s_o - 1.3s_o$ (79)

for one per cent level $0.8s_o - 1.4s_o$

The confidence limits of individual residuals are

for five per cent level $\pm 2.0s$

for one per cent level $\pm 2.7s$ (80)

The weight numbers of corrected coordinates are

$$Q_{xx} = Q_{yy} = \frac{26}{25} + \frac{x^2 + y^2}{50a^2} \tag{81}$$

and the corresponding standard errors

$$s_x = s_y = s_o \sqrt{\frac{26}{25} + \frac{x^2 + y^2}{50a^2}} \tag{82}$$

The distribution of the standard errors within the area $-2a<x<+2a$, $-2a<y<+2a$ is shown in Fig. 9. The root mean square value of the standard errors over the surface 4a x 4a is about 1.06 s_o. The corresponding value for the nine-point solution was found to be 1.1 s_o. No appreciable increase of the accuracy is, consequently, to be expected from the 25-point solution in this respect and in comparison with the nine-point solution. Therefore, it seems sufficient in most cases to use the nine-point solution. Only the reliability of the standard error of unit weight becomes increased through the 25-point solution. In some cases, this fact is of great importance.

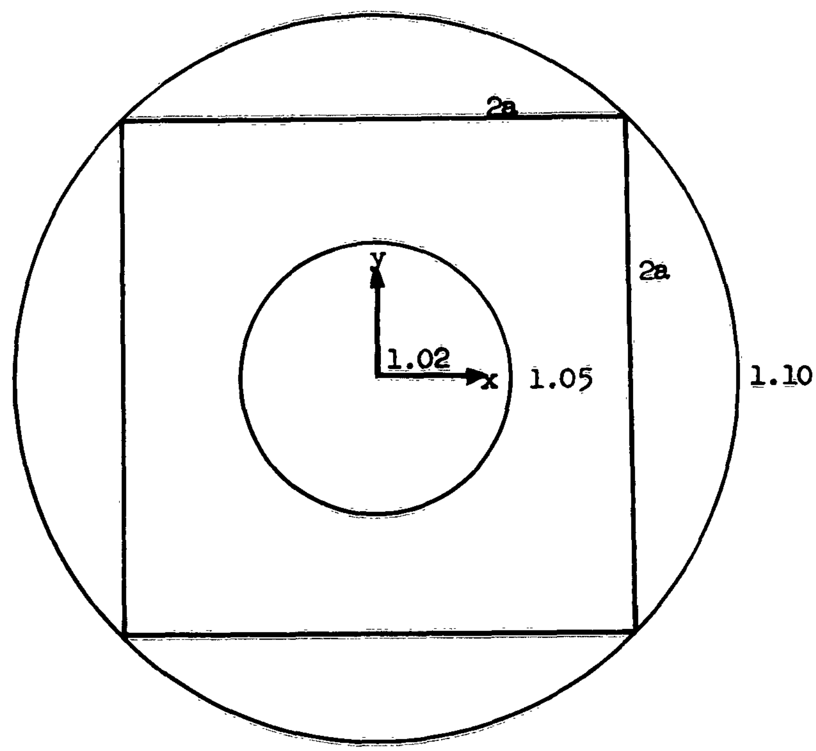


Figure 9. Distribution of standard errors of corrected coordinates; corrections determined from 25 points.

For the practical computations of the adjustment, see forms 1 and 4 (Appendix).

III. GENERAL PRINCIPLES OF THE ADJUSTMENT PROCEDURE, ASSUMING ERRORS IN THE COMPARATOR AND GRID

The assumption made under Section II that the grid coordinates can be regarded as errorless, of course, means a certain approximation. No grid coordinates can be regarded to be completely free from errors since they also have to be measured in a comparator. In general, therefore, the test procedure of a comparator should also take into account the fact that the grid coordinates may be affected with errors. Since these errors originally arose from the measurements in another comparator, they can be assumed to be of the same general character as the errors of the comparator to be investigated.

Neglecting gross errors, the errors to be treated are of regular (systematic)and irregular (random, accidental) character. As above, distinction is made between the regular errors of the measured coordinates caused by the adjustment of the grid in the plate holder of the comparator; i.e., two translations dx_o and dy_o and one rotation of the grid in its plane $d\alpha$, and the regular errors caused by different scales in the x and y directions and lack of orthogonality between the axes, dm_x, dm_y, and $d\beta$ respectively. The regular errors dx_o, dy_o, and $d\alpha$ are of less importance for the accuracy of the comparator or the grid since they can be made very small through careful work and, moreover, can be easily determined. The scale errors and lack of orthogonality have to be expected in the grid as well as in the comparator and must be distinguished. In the adjustment procedures as treated in Section II, a distinction between the errors of the grid and of the comparator cannot be made. The corrections obtained as well as the standard errors of unit weight were referred only to the comparator but will, of course, always contain the errors of the grid as well.

It is possible, however, as will be shown in this section, to distinguish between the regular errors dm_x, dm_y, and $d\beta$ of the comparator and of the grid if the grid is measured in different positions. The method of least squares will be applied for the distinction between the actual errors and for an estimation of irregular errors as standard error of unit weight.

The regular errors of the grid and comparator are denoted dm_{xg}, dm_{yg}, and $d\beta_g$ and dm_{xc}, dm_{yc}, and $d\beta_c$ respectively. The

grid is assumed to be measured in four positions (I, II, III, and IV) and to be rotated in its plane through 100^g (centigrades) clockwise between the different positions. Further, the grid is observed in two ways, up and down, "U" and "D". In U, the grid lines are on the side of the glass toward the operator. In D, the operator views the grid lines through the glass. A change of the focussing of the optical system from the location U is necessary. Under U and D, four positions (I - IV) are assumed and the following combinations are to be regarded: UI, UII, UIII, and UIV as well as DI, DII, DIII, and DIV.

First, the differential formulas will be derived for the different combinations and then general adjustments according to the method of least squares will be made.

5. Grid Up.

a. Position UI. The position UI when the grid point notations and positions agree with Fig. 6 is used as a starting position. In particular, the influences upon the grid coordinates of the regular errors dm_{xg}, dm_{yg}, and $d\alpha_g$ refer to this position.

In accordance with formulas (5) and (6), the differential relations are:

$$dx_{gI} = dx_{ogI} + x_I dm_{xg} - y_I(d\alpha_{gI} + d\beta_g) \qquad (83)$$

$$dy_{gI} = dy_{ogI} + y_I dm_{yg} + x_I d\alpha_{gI} \qquad (84)$$

The subscript I indicates that the term in question refers to position I only. The corresponding differential expressions for the regular errors of the comparator are:

$$dx_c = x dm_{xc} - y d\beta_c \qquad (85)$$

$$dy_c = y dm_{yc} \qquad (86)$$

The terms of the translations and of the common rotation are deleted from the expressions (85) and (86) for obvious reasons.

The discrepancies between the coordinates of the comparator and of the grid are defined as

$$dx = x_{measured} - x_{given} = dx_c - dx_{gI} \qquad (87)$$

$$dy = y_{measured} - y_{given} = dy_c - dy_{gI} \tag{88}$$

The error equations for position UI become:

$$dx_{UI} = x_c dm_{xc} - yd\beta_c - dx_{ogI} - x_I dm_{xg} + y_I(d\alpha_{gI} + d\beta_g) \tag{89}$$

$$dy_{UI} = y_c dm_{yc} - dy_{ogI} - y_I dm_{yg} - x_I d\alpha_{gI} \tag{90}$$

The working correction equations are:

$$v_x = dx_{gI} - dx_c - dx$$

$$v_y = dy_{gI} - dy_c - dy$$

and after substitution of (83) through (86) and some rearrangement:

$$v_{xUI} = dx_{oI} - x_c dm_{xc} + x_I dm_{xg} + y_c d\beta_c - y_I d\beta_g - y_c d\alpha_I - dx_{UI} \tag{91}$$

$$v_{yUI} = dy_{oI} - y_c dm_{yc} + y_I dm_{yg} + x_c d\alpha_I - dy_{UI} \tag{92}$$

In these equations, the two translations and the common rotation are simply denoted dx_{oI}, dy_{oI}, and $d\alpha_I$ since they are specific for position I and since, in principle, it is unimportant whether they are referred to the grid or to the comparator.

From (91) and (92), normal equations can be formed and solved but a distinction between the regular errors of the grid and of the comparator is impossible. At least one more position, II or IV, must be measured and introduced in the computations.

b. Position UII. This position is obtained from position UI after rotation of the grid in its own plane through 100^g clockwise. Figures 10 and 11 show the relations between the coordinate systems x_c, y_c of the comparator and x_c, y_c of the grid respectively.

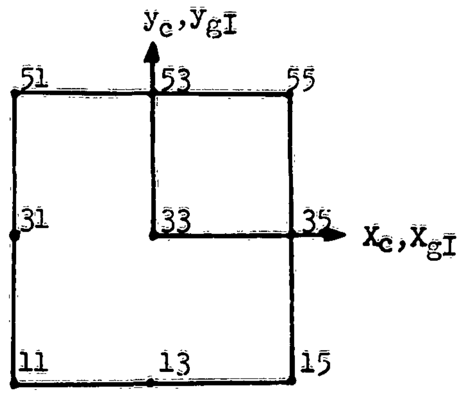


Figure 10. The grid in UI position.

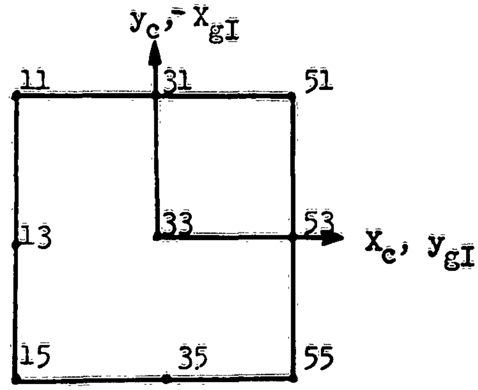


Figure 11. The grid in UII position.

The coordinate system of the comparator is unchanged between the two positions but the x_{gI} corresponds in the UII position to the $-y_c$ and the y_{gI} to the $+x_c$. This means that those parts of the differential formulas (83) and (84) which refer to the regular errors of the grid (dm_{xg}, dm_{yg}, and $d\beta_g$) have to be changed in a corresponding manner in order that the discrepancies between the comparator coordinates x_c, y_c, and the grid coordinates x_{gII}, y_{gII} be correctly interpreted and expressed. The translations dx_{oII}, dy_{oII} and the common rotation $d\alpha_{gII}$ of the grid are specialized for the actual position and have to be expressed accordingly in the differential formulas. The differential formulas of the grid coordinates become:

$$dx_{gII} = dx_{ogII} + y_I dm_{yg} - y_c d\alpha_{gII} \tag{93}$$

$$dy_{gII} = dy_{ogII} - x_I dm_{xg} + y_I d\beta_g + x_c d\alpha_{gII} \tag{94}$$

The differential formulas of the comparator errors are still the expressions (85) and (86). The error equations of the discrepancies between the comparator and grid coordinates of position UII become:

$$dx_{UII} = x_c dm_{xc} - y_c d\beta_c - dx_{ogII} - y_I dm_{yg} + y_c d\alpha_{gII} \tag{95}$$

$$dy_{UII} = y_c dm_{yc} - dy_{ogII} + x_I dm_{xg} - y_I d\beta_g - x_c d\alpha_{gII} \quad (95)$$

The working correction equations become:

$$v_{xUII} = dx_{oII} - x_c dm_{xc} + y_I dm_{yg} + y_c d\beta_c - y_c d\alpha_{II} - dx_{UII} \quad (97)$$

$$v_{yUII} = dy_{oII} - y_c dm_{yc} - x_I dm_{xg} + y_I d\beta_g + x_c d\alpha_{II} - dy_{UII} \quad (98)$$

From the two sets of working correction equations (91), (92) and (97), (98), normal equations can be set up and solved after measurements in a sufficient number of identical grid points. However, it seems suitable for the sake of symmetry to use the measurements in the positions III and IV also and to adjust all the sets of measurements simultaneously. Therefore, the corresponding working correction equations for positions III and IV will be derived next and the normal equations will then be set up and solved.

c. Position UIII. The grid in this position is rotated 200^g in its own plane, clockwise from the position UI. The coordinate relations are shown in Fig. 12.

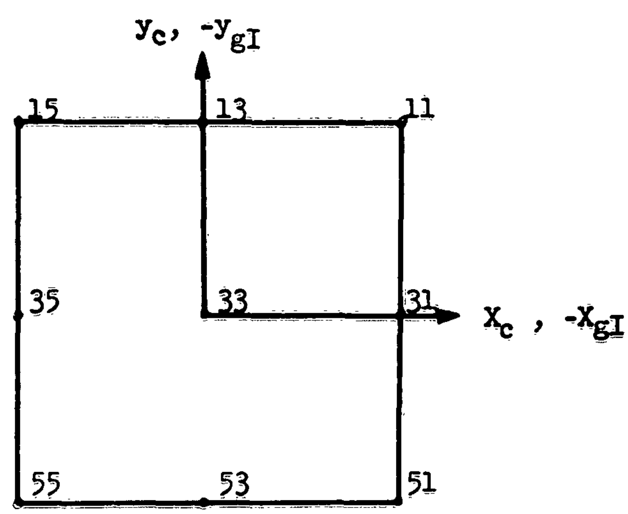


Figure 12. The grid in UIII position.

After similar considerations as above under section 5b, position UII, we find the following working correction equations:

$$v_{xUIII} = dx_{oIII} - x_c dm_{xc} - x_I dm_{xg} + y_c d\beta_c + y_I d\beta_g - y_c d\alpha_{III} - dx_{UIII} \quad (99)$$

$$v_{yUIII} = dy_{oIII} - y_c dm_{yc} - y_I dm_{yg} + x_c d\alpha_{III} - dy_{UIII} \qquad (100)$$

d. Position UIV. The grid in this position is rotated 300^g clockwise in its own plane from the position UI. The coordinate relations are shown in Fig. 13.

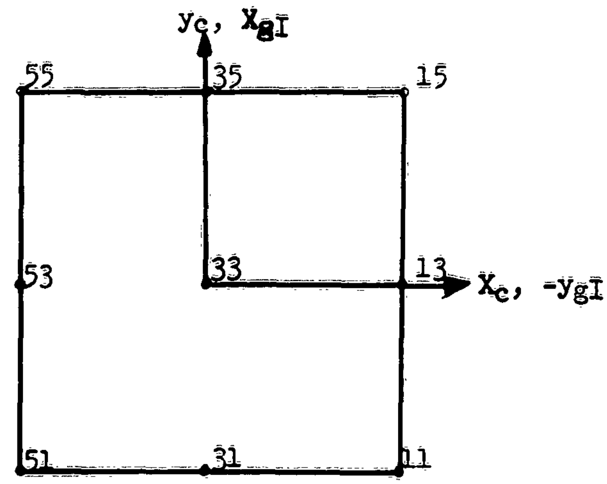


Figure 13. The grid in the UIV position.

After similar considerations as above under Section 5b, position UII, we find the following working correction equations:

$$v_{xUIV} = dx_{oIV} - x_c dm_{xc} - y_I dm_{yg} + y_c d\beta_c - y_c d\alpha_{IV} - dx_{UIV} \qquad (101)$$

$$v_{yUIV} = dy_{oIV} - y_c dm_{yc} + x_I dm_{xg} - y_I d\beta_g + x_c d\alpha_{IV} - dy_{UIV} \qquad (102)$$

e. Joint Adjustment of All Four Sets of Measurements. In summary, the working corrections are for the four U-positions:

$$v_{xUI} = dx_{oI} - x_c dm_{xc} + x_I dm_{xg} + y_c d\beta_c - y_I d\beta_g - y_c d\alpha_I - dx_{UI} \qquad (91)$$

$$v_{yUI} = dy_{oI} - y_c dm_{yc} + y_I dm_{yg} + x_c d\alpha_I - dy_{UI} \qquad (92)$$

$$v_{xUII} = dx_{oII} - x_c dm_{xc} + y_I dm_{yg} + y_c d\beta_c - y_c d\alpha_{II} - dx_{UII} \qquad (97)$$

$$v_{yUII} = dy_{oII} - y_c dm_{yc} - x_I dm_{xg} + y_I d\beta_g + x_c d\alpha_{II} - dy_{UII} \qquad (98)$$

$$v_{xUIII} = dx_{oIII} - x_c dm_{xc} - x_I dm_{xg} + y_c d\beta_c + y_I d\beta_g - y_c d\alpha_{III} - dx_{UIII} \tag{99}$$

$$v_{yUIII} = dy_{oIII} - y_c dm_{yc} - y_I dm_{yg} + x_c d\alpha_{III} - dy_{UIII} \tag{100}$$

$$v_{yUIV} = dx_{oIV} - x_c dm_{xc} - y_I dm_{yg} + y_c d\beta_c - y_c d\alpha_{IV} - dx_{UIV} \tag{101}$$

$$v_{yUIV} = dy_{oIV} - y_c dm_{yc} + x_I dm_{xg} - y_I d\beta_g + x_c d\alpha_{IV} - dy_{UIV} \tag{102}$$

These working correction equations can now be applied to measurements in a suitable number of grid points. The actual grid points are preferably denoted on the glass side, and the recorded coordinates must refer to these notations.

In Tables III and IV, the nine points used in Section II3 above (see also Fig. 6), are assumed to have been measured. The tables show the coefficients of the parameters from the working correction equations. The normal equations are then formed from the tables in the usual manner. See equations (103) through (106).

Table III. Working Correction Equations; Adjustment of Four Sets of Grid Measurements; Location U; Nine Points

	Grid Point	x_c	y_c	x_I	y_I	dx_{oI}	dx_{oII}	dx_{oIII}	dx_{oIV}	dm_{xc}	dm_{xg}	dm_{yg}	$d\beta_c$	$d\beta_g$	$d\alpha_I$	$d\alpha_{II}$	$d\alpha_{III}$	$d\alpha_{IV}$	dx
v_{xUI}	11	-2a	-2a	-2a	-2a	+1				+2a	-2a		-2a	+2a	+2a				$-dx_{11I}$
	13	0	-2a	0	-2a	+1				0	0		-2a	+2a	+2a				$-dx_{13I}$
	15	+2a	-2a	+2a	-2a	+1				-2a	+2a		-2a	+2a	+2a				$-dx_{15I}$
	31	-2a	0	-2a	0	+1				+2a	-2a		0	0	0				$-dx_{31I}$
	33	0	0	0	0	+1				0	0		0	0	0				$-dx_{33I}$
	35	+2a	0	+2a	0	+1				-2a	+2a		0	0	0				$-dx_{35I}$
	51	-2a	+2a	-2a	+2a	+1				+2a	-2a		+2a	-2a	-2a				$-dx_{51I}$
	53	0	+2a	0	+2a	+1				0	0		+2a	-2a	-2a				$-dx_{53I}$
	55	+2a	+2a	+2a	+2a	+1				-2a	+2a		+2a	-2a	-2a				$-dx_{55I}$
v_{xUII}	11	-2a	+2a	-2a	-2a		+1			+2a		-2a	+2a			-2a			$-dx_{11II}$
	13	-2a	0	0	-2a		+1			+2a		-2a	0			0			$-dx_{13II}$
	15	-2a	-2a	+2a	-2a		+1			+2a		-2a	-2a			+2a			$-dx_{15II}$
	31	0	+2a	-2a	0		+1			0		0	+2a			-2a			$-dx_{31II}$
	33	0	0	0	0		+1			0		0	0			0			$-dx_{33II}$
	35	0	-2a	+2a	0		+1			0		0	-2a			+2a			$-dx_{35II}$
	51	+2a	+2a	-2a	+2a		+1			-2a		+2a	+2a			-2a			$-dx_{51II}$
	53	+2a	0	0	+2a		+1			-2a		+2a	0			0			$-dx_{53II}$
	55	+2a	-2a	+2a	+2a		+1			-2a		+2a	-2a			+2a			$-dx_{55II}$

Table III. (continued)																			
	Grid Point	x_c	y_c	x_I	y_I	dx_{oI}	dx_{oII}	dx_{oIII}	dx_{oIV}	dm_{xc}	dm_{xg}	dm_{yg}	$d\beta_c$	$d\beta_g$	$d\alpha_I$	$d\alpha_{II}$	$d\alpha_{III}$	$d\alpha_{IV}$	dx
v_{xUIII}	11	+2a	+2a	-2a	-2a			+1		-2a	+2a		+2a	-2a			-2a		$-dx_{11III}$
	13	0	+2a	0	-2a			+1		0	0		+2a	-2a			-2a		$-dx_{13III}$
	15	-2a	+2a	+2a	-2a			+1		+2a	-2a		+2a	-2a			-2a		$-dx_{15III}$
	31	+2a	0	-2a	0			+1		-2a	+2a		0	0			0		$-dx_{31III}$
	33	0	0	0	0			+1		0	0		0	0			0		$-dx_{33III}$
	35	-2a	0	+2a	0			+1		+2a	-2a		0	0			0		$-dx_{35III}$
	51	+2a	-2a	-2a	+2a			+1		-2a	+2a		-2a	+2a			+2a		$-dx_{51III}$
	53	0	-2a	0	+2a			+1		0	0		-2a	+2a			+2a		$-dx_{53III}$
	55	-2a	-2a	+2a	+2a			+1		+2a	-2a		-2a	+2a			+2a		$-dx_{55III}$
v_{xUIV}	11	+2a	-2a	-2a	-2a				+1	-2a		+2a	-2a					+2a	$-dx_{11IV}$
	13	+2a	0	0	-2a				+1	-2a		+2a	0					0	$-dx_{13IV}$
	15	+2a	+2a	+2a	-2a				+1	-2a		+2a	+2a					-2a	$-dx_{15IV}$
	31	0	-2a	-2a	0				+1	0		0	-2a					+2a	$-dx_{31IV}$
	33	0	0	0	0				+1	0		0	0					0	$-dx_{33IV}$
	35	0	+2a	+2a	0				+1	0		0	+2a					-2a	$-dx_{35IV}$
	51	-2a	-2a	-2a	+2a				+1	+2a		-2a	-2a					+2a	$-dx_{51IV}$
	53	-2a	0	0	+2a				+1	+2a		-2a	0					0	$-dx_{53IV}$
	55	-2a	+2a	+2a	+2a				+1	+2a		-2a	+2a					-2a	$-dx_{55IV}$

The x_I- and y_I-coordinates correspond to the coordinates of grid points in position UI and are in this position identical with the comparator-coordinates.

Table IV. Working Correction Equations: Adjustment of Four Sets of Grid Coordinates: Location U; Nine Points

	Grid Point	x_c	y_c	x_I	y_I	dy_{oI}	dy_{oII}	dy_{oIII}	dy_{oIV}	dm_{yc}	dm_{xg}	dm_{yg}	$d\theta_g$	$d\alpha_I$	$d\alpha_{II}$	$d\alpha_{III}$	$d\alpha_{IV}$	dy
v_{yUI}	11	-2a	-2a	-2a	-2a	+1				+2a		-2a		-2a				$-dy_{11I}$
	13	0	-2a	0	-2a	+1				+2a		-2a		0				$-dy_{13I}$
	15	+2a	-2a	+2a	-2a	+1				+2a		-2a		+2a				$-dy_{15I}$
	31	-2a	0	-2a	0	+1				0		0		-2a				$-dy_{31I}$
	33	0	0	0	0	+1				0		0		0				$-dy_{33I}$
	35	+2a	0	+2a	0	+1				0		0		+2a				$-dy_{35I}$
	51	-2a	+2a	-2a	+2a	+1				-2a		+2a		-2a				$-dy_{51I}$
	53	0	+2a	0	+2a	+1				-2a		+2a		0				$-dy_{53I}$
	55	+2a	+2a	+2a	+2a	+1				-2a		+2a		+2a				$-dy_{55I}$
v_{yUII}	11	-2a	+2a	-2a	-2a		+1			-2a	+2a		-2a		-2a			$-dy_{11II}$
	13	-2a	0	0	-2a		+1			0	0		-2a		-2a			$-dy_{13II}$
	15	-2a	-2a	+2a	-2a		+1			+2a	-2a		-2a		-2a			$-dy_{15II}$
	31	0	+2a	-2a	0		+1			0	+2a		0		0			$-dy_{31II}$
	33	0	0	0	0		+1			0	0		0		0			$-dy_{33II}$
	35	0	-2a	+2a	0		+1			+2a	-2a		0		0			$-dy_{35II}$
	51	+2a	+2a	-2a	+2a		+1			-2a	+2a		+2a		+2a			$-dy_{51II}$
	53	+2a	0	0	+2a		+1			0	0		+2a		+2a			$-dy_{53II}$
	55	+2a	-2a	+2a	+2a		+1			+2a	-2a		+2a		+2a			$-dy_{55II}$

Table IV. (continued)

	Grid Point	x_c	y_c	x_I	y_I	dy_{oI}	dy_{oII}	dy_{oIII}	dy_{oIV}	dm_{yc}	dm_{xg}	dm_{yg}	$d\beta_g$	$d\alpha_I$	$d\alpha_{II}$	$d\alpha_{III}$	$d\alpha_{IV}$	dy
vy_{UIII}	11	+2a	+2a	-2a	-2a			+1		-2a		+2a				+2a		$-dy_{11III}$
	13	0	+2a	0	-2a			+1		-2a		+2a				0		$-dy_{13III}$
	15	-2a	+2a	+2a	-2a			+1		-2a		+2a				-2a		$-dy_{15III}$
	31	+2a	0	-2a	0			+1		0		0				+2a		$-dy_{31III}$
	33	0	0	0	0			+1		0		0				0		$-dy_{33III}$
	35	-2a	0	+2a	0			+1		0		0				-2a		$-dy_{35III}$
	51	+2a	-2a	-2a	+2a			+1		+2a		-2a				+2a		$-dy_{51III}$
	53	0	-2a	0	+2a			+1		+2a		-2a				0		$-dy_{53III}$
	55	-2a	-2a	+2a	+2a			+1		+2a		-2a				-2a		$-dy_{55III}$
vy_{UIV}	11	+2a	-2a	-2a	-2a				+1	+2a	-2a		+2a				+2a	$-dy_{11IV}$
	13	+2a	0	0	-2a				+1	0	0		+2a				+2a	$-dy_{13IV}$
	15	+2a	+2a	+2a	-2a				+1	-2a	+2a		+2a				+2a	$-dy_{15IV}$
	31	0	-2a	-2a	0				+1	+2a	-2a		0				0	$-dy_{31IV}$
	33	0	0	0	0				+1	0	0		0				0	$-dy_{33IV}$
	35	0	+2a	+2a	0				+1	-2a	+2a		0				0	$-dy_{35IV}$
	51	-2a	-2a	-2a	+2a				+1	+2a	-2a		-2a				-2a	$-dy_{51IV}$
	53	-2a	0	0	+2a				+1	0	0		-2a				-2a	$-dy_{53IV}$
	55	-2a	+2a	+2a	+2a				+1	-2a	+2a		-2a				-2a	$-dy_{55IV}$

$$
\begin{aligned}
9dx_{oI} - [dx]_I &= 0 \\
9dx_{oII} - [dx]_{II} &= 0 \\
9dx_{oIII} - [dx]_{III} &= 0 \\
9dx_{oIV} - [dx]_{IV} &= 0
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
9dy_{oI} - [dy]_I &= 0 \\
9dy_{oII} - [dy]_{II} &= 0 \\
9dy_{oIII} - [dy]_{III} &= 0 \\
9dy_{oIV} - [dy]_{IV} &= 0
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
96a^2dm_{xc} - 48a^2dm_{xg} - 48a^2dm_{yg} + 2aT_1 &= 0 \\
96a^2dm_{yc} - 48a^2dm_{xg} - 48a^2dm_{yg} + 2aT_2 &= 0 \\
-48a^2dm_{xc} - 48a^2dm_{yc} + 96a^2dm_{xg} + 2aT_3 &= 0 \\
-48a^2dm_{xc} - 48a^2dm_{yc} + 96a^2dm_{yg} + 2aT_4 &= 0
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
&96a^2d\beta_c - 48a^2d\beta_g - 24a^2d\alpha_I - 24a^2d\alpha_{II} - 24a^2d\alpha_{III} - 24a^2d\alpha_{IV} + \\
&\qquad + 2aT_5 = 0 \\
&-48a^2d\beta_c + 96a^2d\beta_g + 24a^2d\alpha_I + 24a^2d\alpha_{II} + 24a^2d\alpha_{III} + 24a^2d\alpha_{IV} + \\
&\qquad + 2aT_6 = 0 \\
&-24a^2d\beta_c + 24a^2d\beta_g + 48a^2d\alpha_I + 2aT_7 = 0 \\
&-24a^2d\beta_c + 24a^2d\beta_g + 48a^2d\alpha_{II} + 2aT_8 = 0 \\
&-24a^2d\beta_c + 24a^2d\beta_g + 48a^2d\alpha_{III} + 2aT_9 = 0 \\
&-24a^2d\beta_c + 24a^2d\beta_g + 48a^2d\alpha_{IV} + 2aT_{10} = 0
\end{aligned}
\tag{106}
$$

There are four groups of normal equations which are mutually independent. The symbols $T_1 - T_{10}$ are defined as follows:

$$T_1 = (-dx_{11} + dx_{15} - dx_{31} + dx_{35} - dx_{51} + dx_{55})_I +$$

$$+(-dx_{11} - dx_{13} - dx_{15} + dx_{51} + dx_{53} + dx_{55})_{II} +$$

$$+(dx_{11} - dx_{15} + dx_{31} - dx_{35} + dx_{51} - dx_{55})_{III} +$$

$$+(dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55})_{IV} \tag{107}$$

$$T_2 = (-dy_{11} - dy_{13} - dy_{15} + dy_{51} + dy_{53} + dy_{55})_I +$$

$$+(dy_{11} - dy_{15} + dy_{31} - dy_{35} + dy_{51} - dy_{55})_{II} +$$

$$+(dy_{11} + dy_{13} + dy_{15} - dy_{51} - dy_{53} - dy_{55})_{III} +$$

$$+(-dy_{11} + dy_{15} - dy_{31} + dy_{35} - dy_{51} + dy_{55})_{IV} \tag{108}$$

$$T_3 = (dx_{11} - dx_{15} + dx_{31} - dx_{35} + dx_{51} - dx_{55})_I +$$

$$+(-dx_{11} + dx_{15} - dx_{31} + dx_{35} - dx_{51} + dx_{55})_{III} +$$

$$+(-dy_{11} + dy_{15} - dy_{31} + dy_{35} - dy_{51} + dy_{55})_{II} +$$

$$+(dy_{11} - dy_{15} + dy_{31} - dy_{35} + dy_{51} - dy_{55})_{IV} \tag{109}$$

$$T_4 = (dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55})_{II} +$$

$$+(-dx_{11} - dx_{13} - dx_{15} + dx_{51} + dx_{53} + dx_{55})_{IV} +$$

$$+(dy_{11} + dy_{13} + dy_{15} - dy_{51} - dy_{53} - dy_{55})_I +$$

$$+(-dy_{11} - dy_{13} - dy_{15} + dy_{51} + dy_{53} + dy_{55})_{III} \tag{110}$$

$$T_5 = (dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55})_{I} +$$

$$+(-dx_{11} + dx_{15} - dx_{31} + dx_{35} - dx_{51} + dx_{55})_{II} +$$

$$+(-dx_{11} - dx_{13} - dx_{15} + dx_{51} + dx_{53} + dx_{55})_{III} +$$

$$+(dx_{11} - dx_{15} + dx_{31} - dx_{35} + dx_{51} - dx_{55})_{IV} \qquad (111)$$

$$T_6 = (-dx_{11} - dx_{13} - dx_{15} + dx_{51} + dx_{53} + dx_{55})_{I} +$$

$$+(dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55})_{III} +$$

$$+(dy_{11} + dy_{13} + dy_{15} - dy_{51} - dy_{53} - dy_{55})_{II} +$$

$$+(-dy_{11} - dy_{13} - dy_{15} + dy_{51} + dy_{53} + dy_{55})_{IV} \qquad (112)$$

$$T_7 = (-dx_{11} - dx_{13} - dx_{15} + dx_{51} + dx_{53} + dx_{55})_{I} +$$

$$+(dy_{11} - dy_{15} + dy_{31} - dy_{35} + dy_{51} - dy_{55})_{I} \qquad (113)$$

$$T_8 = (dx_{11} - dx_{15} + dx_{31} - dx_{35} + dx_{51} - dx_{55})_{II} +$$

$$+(dy_{11} + dy_{13} + dy_{15} - dy_{51} - dy_{53} - dy_{55})_{II} \qquad (114)$$

$$T_9 = (dx_{11} + dx_{13} + dx_{15} - dx_{51} - dx_{53} - dx_{55})_{III} +$$

$$+(-dy_{11} + dy_{15} - dy_{31} + dy_{35} - dy_{51} + dy_{55})_{III} \qquad (115)$$

$$T_{10} = (-dx_{11} + dx_{15} - dx_{31} + dx_{35} - dx_{51} + dx_{55})_{IV} +$$

$$+(-dy_{11} - dy_{13} - dy_{15} + dy_{51} + dy_{53} + dy_{55})_{IV} \qquad (116)$$

The solutions of the normal equation systems are:

$$dx_{cI} = \frac{[dx]_I}{9} \qquad dy_{oI} = \frac{[dy]_I}{9}$$

$$dx_{oII} = \frac{[dx]_{II}}{9} \qquad dy_{oII} = \frac{[dy]_{II}}{9}$$

$$dx_{oIII} = \frac{[dx]_{III}}{9} \qquad dy_{oIII} = \frac{[dy]_{III}}{9}$$

$$dx_{oIV} = \frac{[dx]_{IV}}{9} \qquad dy_{oIV} = \frac{[dy]_{IV}}{9} \tag{117}$$

$$dm_{xc} - dm_{yc} = \frac{T_2 - T_1}{48a}$$

$$dm_{xg} - dm_{yg} = \frac{T_4 - T_3}{48a} \tag{118}$$

$$d\beta_c = -\frac{2T_5 + T_7 + T_8 + T_9 + T_{10}}{48a}$$

$$d\beta_g = \frac{-2T_6 + T_7 + T_8 + T_9 + T_{10}}{48a} \tag{119}$$

$$d\alpha_I = \frac{1}{48a}(-T_5 + T_6 - 3T_7 - T_8 - T_9 - T_{10})$$

$$d\alpha_{II} = \frac{1}{48a}(-T_5 + T_6 - T_7 - 3T_8 - T_9 - T_{10})$$

$$d\alpha_{III} = \frac{1}{48a}(-T_5 + T_6 - T_7 - T_8 - 3T_9 - T_{10})$$

$$d\alpha_{IV} = \frac{1}{48a}(-T_5 + T_6 - T_7 - T_8 - T_9 - 3T_{10}) \tag{120}$$

The following can be obtained from the normal equations:

$$dm_{xc} - dm_{xg} = \frac{-3T_1 + T_2 + 3T_3 - T_4}{192a}$$

$$dm_{yc} - dm_{yg} = \frac{T_1 - 3T_2 - T_3 + 3T_4}{192a} \qquad (121)$$

No absolute scale correction can be obtained from the solution of the normal equations. This is, of course, to be expected since neither the grid nor the comparator is assumed to have given absolute dimensions. Only the differences between the scale corrections can be determined. It is also impossible to determine the accuracy in its real sense since no absolute data are given. Only the precision can be determined from repeated measurements of the grid points in different locations and positions. In order to obtain a better determination of the precision than from repeated settings in certain points only, grid measurements in two suitable positions are used. Such positions should be chosen where the possible regular errors of the grid and of the comparator become compensated as far as possible. The simplest method would be to measure the grid coordinates in one position twice and as independently as possible, preferably with a reset of the grid in the plate holder between the two sets of measurements. The computation should be performed as an adjustment using three parameters, two translations and one common rotation. The adjustment should be performed as if one of the sets of measurements were the given values of the coordinates. But this procedure may give misleading results due to the possible correlation between the two sets of measurements. Therefore, it seems to be more suitable to use the positions I and III or II and IV for the determination of the basic quality of the measurements. Since this quality will be obtained from adjustments of indirect observations and as the results of solutions of normal equations, it may be defined as standard deviation of unit weight. There is a clear difference between this concept and the standard error of unit weight, which requires that the observed discrepancies refer to given data which can be regarded as errorless at least in comparison with the measurements.

The procedure shown above in sections II3 and II4 can be used for the determination of the sum of the squares of the residuals and the standard deviation of unit weight. It must be noted, however, that correlations may occur which result in a standard deviation of unit weight value that is too low.

The reliability of the computed corrections can be determined in the usual way with the aid of weight and correlation numbers and the basic standard error or standard deviation of unit weight of the observations. For the computation of the weight numbers of the expressions in (117), the definition of such numbers can directly be applied. All of them become $\frac{1}{9}$. The weight numbers of the expressions (118) through (120) are preferably computed via the weight and correlation numbers of the T-expressions (107) through (116). These are shown in the following weight matrix (Table V).

Table V. Weight and Correlation Numbers

	Q_{T1}	Q_{T2}	Q_{T3}	Q_{T4}	Q_{T5}	Q_{T6}	Q_{T7}	Q_{T8}	Q_{T9}	Q_{T10}
Q_{T1}	24	0	-12	-12	0	0	0	0	0	0
Q_{T2}		24	-12	-12	0	0	0	0	0	0
Q_{T3}			24	0	0	0	0	0	0	0
Q_{T4}				24	0	0	0	0	0	0
Q_{T5}					24	-12	-6	-6	-6	-6
Q_{T6}						24	+6	+6	+6	+6
Q_{T7}							12	0	0	0
Q_{T8}								12	0	0
Q_{T9}									12	0
Q_{T10}										12

The weight and correlations numbers of the expressions (118) through (120) can now be computed.

We find, for instance:

$$Q_{(m_{xc} - m_{yc})(m_{xc} - m_{yc})} = \frac{Q_{T1T1} + Q_{T2T2}}{(48a)^2} = \frac{1}{48a^2} \quad (122)$$

$$Q_{(m_{xg} - m_{yg})(m_{xg} - m_{yg})} = \frac{1}{48a^2} \quad (123)$$

$$Q_{\beta c\beta c} = \frac{1}{(48a)^2} (4Q_{T5T5} + Q_{T7T7} + Q_{T8T8} + Q_{T9T9} + Q_{T10T10} +$$

$$+ 4Q_{T5T7} + 4Q_{T5T8} + 4Q_{T5T9} + 4Q_{T5T10} + 2Q_{T7T8} +$$

$$+ 2Q_{T7T9} + 2Q_{T7T10} + 2Q_{T8T9} + 2Q_{T8T10} + 2Q_{T9T10}) = \frac{1}{48a^2}$$

$$Q_{\beta c\beta c} = Q_{\beta g\beta g} = \frac{1}{48a^2} \quad (124)$$

These are the most important weight numbers. For further determinations of weight and correlation numbers, the same principles can be used. When the weight and correlation numbers have been determined, the quality of all functions of the adjusted data can be determined.

6. Grid Down. From Section III5, it is clear that it is sufficient to measure the grid in one location in order to determine the actual regular errors of the grid and of the comparator. The same procedure used for the U-location can be applied to the D-location. One of the D-positions is chosen as origin, to which the regular errors of the grid are referred. If the regular errors are to be referred to one of the U-positions, the relation between this position and the D-positions must be determined. This can be done in a way similar to that of Section III5; i.e., regarding the change of the coordinate directions from one position to another.

Since the procedure is very similar to what has been treated above under Section III5, a detailed derivation of formulas, etc., is not given.

In the practical examples of comparator tests as shown in Section IV, some determinations of regular errors from measurements of the same grid in U and D-positions will be made.

7. Determination of the Absolute Scale of a Comparator. So far, under point III, only the differences in scale of the two coordinate directions of the comparator and of the grid have been determined. The absolute scale remains to be determined. This must be done from the comparison of a given scale of high accuracy with the scales of the comparator. Since the differences have been determined, it is sufficient to determine one of the comparator scales only.

In n points, the x-coordinates of an accurate scale have been measured in the comparator. One translation dx_o and one scale error dm_x are assumed as regular errors of the measured coordinates. The influences upon the measured coordinates from these two parameters are expressed by the differential formula

$$dx = dx_o + xdm_x \tag{125}$$

The measured coordinates x_m are compared with the given coordinates x_g, and the discrepancies are defined as

$$dx = x_m - x_g \tag{126}$$

From (125), the correction formula is obtained as

$$dx = x_m - x_g = -dx_o - xdm_x \tag{127}$$

Since redundant measurements are made, the expression (127) is written as a working correction equation:

$$v = -dx_o - xdm_x - dx \tag{128}$$

For the most convenient treatment of the adjustment problem, it is suitable to use the point of gravity as origin. This point is defined through

$$\frac{[x]}{n}$$

The coordinates, referred to the point of gravity, are defined

$$X = x - \frac{[x]}{n} \tag{129}$$

Hence $[X] = 0$

The working correction equation becomes

$$v = -dx_o - Xdm_x - dx \tag{130}$$

For the n points, the normal equations become

$$\begin{aligned} n\,dx_o + [dx] &= 0 \\ [XX]dm_x + [Xdx] &= 0 \end{aligned} \qquad (131)$$

The sum [vv] can be found from the equation:

$$[vv] = [dxdx] + [dx]dx_o + [Xdx]dm_x \qquad (132)$$

The solution of the normal equations is simply:

$$dx_o = -\frac{[dx]}{n} \qquad (133)$$

$$dm_x = -\frac{[Xdx]}{[XX]} \qquad (134)$$

$$[vv] = [dxdx] - \frac{[dx]^2}{n} - \frac{[Xdx]^2}{[XX]} \qquad (135)$$

The standard error of unit weight is

$$s_o = \sqrt{\frac{[vv]}{n-2}} \qquad (136)$$

The weight numbers are

$$Q_{x_o x_o} = \frac{1}{n} \qquad (137)$$

$$Q_{m_x m_x} = \frac{1}{[XX]} \qquad (138)$$

The correlation number is zero.

Corrections to arbitrary points can be computed from (127) or (130). The weight number of a corrected coordinate is, according to (130):

$$Q_{xx} = \frac{1}{n} + \frac{X^2}{[XX]} + 1 \qquad (139)$$

and the corresponding standard error

$$s_x = s_o\sqrt{\frac{n+1}{n} + \frac{X^2}{[XX]}} \qquad (140)$$

The root mean square value of the standard errors within a certain interval $X_1 - X_2$ can be found from the expression

$$M^2_{s_X} = \frac{s_o^2}{(X_2 - X_1)} \int_{X_1}^{X_2} \left(\frac{n+1}{n} + \frac{X^2}{[XX]} \right) dX \tag{141}$$

After integration, we find

$$M^2_{s_X} = s_o^2 \left(\frac{n+1}{n} + \frac{X_2^3 - X_1^3}{3[XX]\,(X_2 - X_1)} \right) \tag{142}$$

The standard error of the standard error of unit weight is found from

$$s_{s_o} = \frac{s_o}{\sqrt{2(n-2)}}$$

The confidence limits can be determined for actual cases with respect to the number of redundant observations (degrees of freedom) and the confidence level (usually 5 per cent). A practical example of the determination of the absolute scale of a comparator is shown in Section IV.

If more regular errors are present than those which were assumed in the calculation above, it is desirable to determine the individual residuals after the adjustment with the aid of expression (130) and to plot a diagram. Further, a test of the normal distribution is always very useful, provided that enough residuals are available.

IV. PRACTICAL EXAMPLES

It is desirable to test the procedures and formulas derived above in order to check the expressions obtained and to determine the order of magnitude of the basic precision and accuracy. In fact, such tests must be performed comprehensively in many different instruments with different grids and by many operators in order to determine the basic factors upon which the determination of tolerances, etc., must be founded.

A series of practical tests will be shown below. These tests have been made primarily for the purposes mentioned above but all of them are not to be regarded as representative for the highest precision and accuracy which can be obtained from high-precision comparators. The operator was not particularly well-trained or

experienced in some cases.

First, the results of a series of measurements for the determination of the precision of image coordinate measurements in a single image comparator will be shown. The focussing of the optical system was intentionally changed between the series in order to see the influence upon the precision of this variation.

Further, the formulas derived above will be tested with adjustments of discrepancies, caused by introduced artificial regular errors in the grid and comparator coordinates. Then, series of practical measurements in a single image comparator will be treated in different ways. The absolute scale determination of a comparator will be shown in detail in accordance with practical measurements from the National Bureau of Standards. Finally, the results of practical measurements in some modern stereocomparators will be shown and discussed.

8. Determination of Precision of Image Coordinate Measurements in a Single Image Comparator under Different Conditions. The procedure is very simple. The settings of the measuring mark in a grid point were repeated at least 20 times, and the corresponding x and y-coordinates were recorded. Three series with different focussing were measured, and the standard deviation of one measurement was computed for each series according to well-known procedures. The detailed readings may be of little interest in this connection. Therefore, only one series of the measurements will be completely shown while for the others only the results will be given. (See Table VI).

From Table VI, we find:

$$x_{average} = 84.0638 \text{ mm}$$

$$y_{average} = 344.8127 \text{ mm}$$

$$[v_x v_x] = 159.24 \text{ micron}^2$$

$$[v_y v_y] = 128.69 \text{ micron}^2$$

Hence, the standard deviations of one x- and y-measurement, respectively:

$$s_x = \sqrt{\frac{159.24}{20}} = 2.8 \text{ microns}$$

$$s_y = \sqrt{\frac{128.69}{20}} = 2.5 \text{ microns}$$

Table VI. Determination of Precision of Image Coordinate Measurements in a Single Image Comparator; Bad Focussing				
Setting nr	x (mm)	y (mm)	v_x (microns)	v_y (microns)
1	84.062	344.811	+1.8	+1.7
2	61	07	+2.8	+5.7
3	63	09	+0.8	+3.7
4	61	15	+2.8	-2.3
5	67	14	-3.2	-1.3
6	62	09	+1.8	+3.7
7	60	12	+3.8	+0.7
8	57	11	+6.8	+1.7
9	67	14	-3.2	-1.3
10	64	15	-0.2	-2.3
11	63	13	+0.8	-0.3
12	60	13	+3.8	-0.3
13	66	12	-2.2	+0.7
14	65	12	-1.2	+0.7
15	66	17	-2.2	-4.3
16	67	15	-3.2	-2.3
17	66	13	-2.2	-0.3
18	65	16	-1.2	-3.3
19	61	10	+2.8	+2.7
20	65	13	-1.2	-0.3
21	62	15	+1.8	-2.3

From measurements of the same type with an improved, but not the best focussing, the following results were obtained:

$$s_x = 1.7 \text{ microns}$$

$$s_y = 1.9 \text{ microns}$$

Finally, the best focussing was introduced, and the measurements were repeated. The following standard deviations were found:

$$s_x = 1.8 \text{ microns}$$

$$s_y = 1.0 \text{ microns}$$

Consequently, a clear relation between focussing and standard deviations was found. It is evidently of great importance to adjust the focussing carefully in order to obtain a high precision of the measurements. This may be rather self explanatory but so far no numerical information has been given on this subject. Much more research should be devoted to this important question. The degree of sharpness should be determined in some way, possibly with the aid of resolving power targets or some form of optical transfer tests in order to determine the correlation between sharpness and standard deviation. Similar procedures should, moreover, be applied to the imaging system of cameras.

In summary, the standard deviation of the image coordinate measurements (one measurement in x or in y) was for the actual instrument, grid, and operator found to be 1.4 micron as an average.

If two settings always are to be used and the average of the results is to be computed and used for further computations the standard deviation of this average is found by dividing the figure 1.4 microns by $\sqrt{2}$. The standard deviation of the average of two measurements in the actual instrument, in the actual grid, and by the actual operator is, consequently,

$$s_x = s_y = 1.0 \text{ micron}$$

This is evidently the lower limit for the standard error of unit weight of the image coordinate measurements, expressing the accuracy of the procedure. In other words, there is little reason for trying to increase the accuracy beyond the obtained value of the precision.

Another test of the precision was also obtained from repeated series of measurements in sets of nine points. In each point, only one setting was made in each series. A grid was measured twice in two locations, grid lines up and down respectively. From the differences between the two sets of image coordinates in each location, the standard deviation of one setting and in the average of two settings was computed. As an average, the value 0.8 micron was found, which agrees rather well with the value shown above. It is also an indication of the very good stability of the instrument.

9. Tests of Formulas and Procedures with the Aid of Artificial Errors. The most reliable and effective way to test complicated formula systems for adjustments and determination of regular errors is to use fictive coordinates containing known artificial errors. Some examples of this procedure will be given here before the examples of application of the procedures to

real grid measurements in a single comparator.

The first step in the actual test procedure is to assume certain regular errors in the grid and in the comparator and then to apply the adjustment formulas to the computed discrepancies between the grid and comparator coordinates. The results must agree with the introduced regular errors, and the standard error of unit weight must become zero. The procedure will become clear from the example below. We assume nine regularly located points according to Figure 14. The correct assumed coordinates of the points are indicated together with the notations of the points.

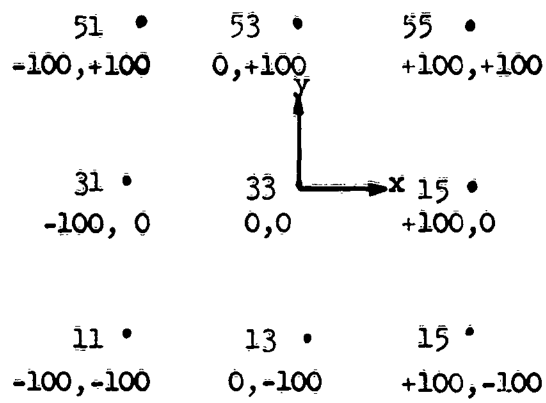


Figure 14. Notations and Coordinates of the Assumed Errorless Grid Points.

Next, we assume the following regular errors to affect the grid coordinates:

$$dx_o = +0.010 \text{ mm}$$
$$dy_o = +0.010 \text{ mm}$$
$$dm_{xg} = +0.000020$$
$$dm_{yg} = -0.000020$$
$$d\alpha_g = +0.000030$$
$$d\beta_g = +0.000010$$

The comparator is assumed to be affected by the following regular errors:

$$dm_{xc} = -0.000030$$

$$dm_{yc} = -0.000050$$

$$d\beta_c = +0.000030$$

The influences upon the coordinates of the grid and the comparator from these regular errors are next found from the differential formulas (5) and (6).

$$dx = dx_o + x dm_x - y(d\alpha + d\beta)$$

$$dy = dy_o + x d\alpha + y\, dm_y$$

In the following Tables VII through XI, the corresponding comparator and grid coordinates are shown. The grid position shown in Fig. 14 is denoted UI. The positions UII, UIII, and UIV are obtained after rotations of the grid through 100^g, 200^g, and 300^g respectively.

Table VII. Comparator Coordinates

Point	x (mm)	y (mm)
11	-99.994	-99.995
13	0.003	-99.995
15	100.000	-99.995
31	-99.997	0.000
33	0.000	0.000
35	99.997	0.000
51	-100.000	99.995
53	-0.003	99.995
55	99.994	99.995

Table VIII. Grid Coordinates Position UI

Grid Point	x (mm)	y (mm)
11	-99.988	-99.991
13	0.014	-99.988
15	100.016	-99.985
31	-99.992	0.007
33	0.010	0.010
35	100.012	0.013
51	-99.996	100.005
53	0.006	100.008
55	100.008	100.011

Table IX. Grid Coordinates			
Position UII			
Grid Point	x (mm)	y (mm)	Comp. Point
11	-99.991	99.988	51
13	-99.988	-0.014	31
15	-99.985	-100.016	11
31	0.007	99.992	53
33	0.010	-0.010	33
35	0.013	-100.012	13
51	100.005	99.996	55
53	100.008	-0.006	35
55	100.011	-100.008	15

Table X. Grid Coordinates			
Position UIII			
Grid Point	x (mm)	y (mm)	Comp. Point
11	99.988	99.991	55
13	-0.014	99.988	53
15	-100.016	99.985	51
31	99.992	-0.007	35
33	-0.010	-0.010	33
35	-100.012	-0.013	31
51	99.996	-100.005	15
53	-0.006	-100.008	13
55	-100.008	-100.011	11

Table XI. Grid Coordinates			
Position UIV			
Grid Point	x (mm)	y (mm)	Comp. Point
11	99.991	-99.988	15
13	99.988	+0.014	35
15	99.985	100.016	55
31	-0.007	-99.992	13
33	-0.010	0.010	33
35	-0.013	100.012	53
51	-100.005	-99.996	11
53	-100.008	0.006	31
55	-100.011	100.008	51

The discrepancies to be adjusted are found from each grid position in subtracting the actual grid coordinates from the corresponding comparator coordinates according to Table VII. The computations can then be made in form 3 or forms 5 through 7 (Appendix). If the computations are made according to form 3, the obtained corrections will become combinations of the introduced errors from the grid and the comparator. It is possible, however, to distinguish between the influence of the errors with elementary procedures, particularly if the measurements of the grid had been made also in the D-location (grid lines

down). Since this procedure may be of special interest to practice in many cases, it will be treated here and the results will immediately be tested. Next, the grid coordinates will be shown in the four D-positions, see Tables XII through XV. Fig. 15 shows the positions of the grid points after the turning upside-down of the grid and for position DI.

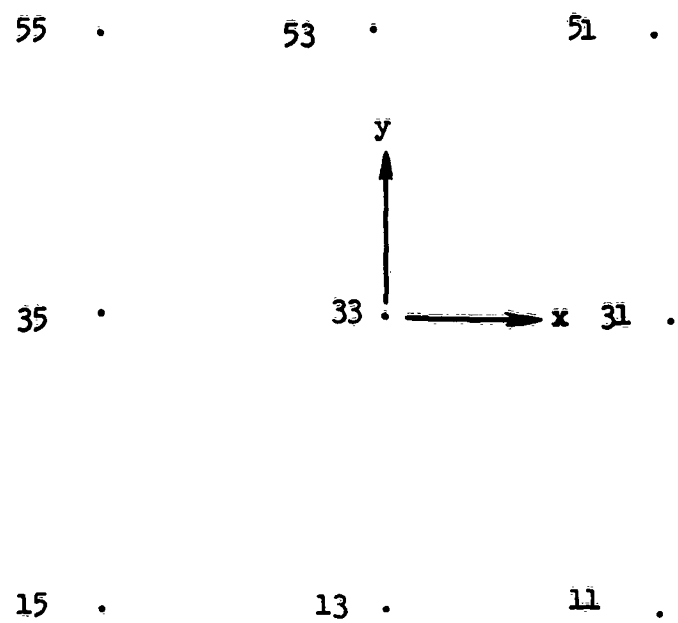


Figure 15. Grid points in the DI-position.

Table XII. Grid Coordinates Position DI

Grid Point	x (mm)	y (mm)	Comp. Point
11	99.988	-99.991	15
13	-0.014	-99.988	13
15	-100.016	-99.985	11
31	99.992	0.007	35
33	-0.010	0.010	33
35	-100.012	0.013	31
51	99.996	100.005	55
53	-0.006	100.008	53
55	-100.008	100.011	51

Table XIII. Grid Coordinates Position DII

Grid Point	x (mm)	y (mm)	Comp. Point
11	-99.991	-99.988	11
31	0.007	-99.992	13
51	100.005	-99.996	15
13	-99.988	0.014	31
33	0.010	0.010	33
53	100.008	0.006	35
15	-99.985	100.016	51
35	0.013	100.012	53
55	100.011	100.008	55

Table XIV. Grid Coordinates Position DIII			
Grid Point	x (mm)	y (mm)	Comp. point
51	-99.996	-100.005	11
53	0.006	-100.008	13
55	100.008	-100.011	15
31	-99.992	-0.007	31
33	0.010	-0.010	33
35	100.012	-0.013	35
11	-99.988	99.991	51
13	0.014	99.988	53
15	100.016	99.985	55

Table XV. Grid Coordinates Position DIV			
Grid Point	x (mm)	y (mm)	Comp. Point
55	-100.011	-100.008	11
35	-0.013	-100.012	13
15	99.985	-100.016	15
53	-100.008	-0.006	31
33	-0.010	-0.010	33
13	99.988	-0.014	35
51	-100.005	99.996	51
31	-0.007	99.992	53
11	99.991	99.988	55

The results of the computations according to form 3 and concerning the regular errors dm_x, dm_y, and $d\beta$ are summarized below in Table XVI.

Table XVI. The Corrections dm_x, dm_y, and $d\beta$ from the Positions UI-UIV and DI-DIV							
Position	dm_x	dm_y	$d\beta$	Position	dm_x	dm_y	$d\beta$
UI	-0.000050	-0.000030	-0.000020	DI	-0.000050	-0.000030	-0.000040
UII	-0.000010	-0.000070	-0.000040	DII	-0.000010	-0.000070	-0.000020
UIII	-0.000050	-0.000030	-0.000020	DIII	-0.000050	-0.000030	-0.000040
UIV	-0.000010	-0.000070	-0.000040	DIV	-0.000010	-0.000070	-0.000020

The corrections of Table XVI are obviously combinations of the errors from the grid and from the comparator. The procedure of distinguishing between the two sources of the errors can be derived as follows:

a. The Scale Errors. The scale errors of the comparator are denoted dm_{xc}, dm_{yc} and of the grid in position UI dm_{xg}, dm_{yg}.

Since the differences between the comparator and the grid coordinates are used in the adjustment procedure, the obtained corrections must consist of the differences between grid and the comparator scale errors. It must be noted that corrections and errors are opposite each other concerning the signs. The coordinate system of the comparator is fixed during all series of measurements but the grid system is rotated as described above. From this fact, the following

relations are found between the corrections from the normal equations and the errors of the comparator and the grid:

$$dm_{xUI} = dm_{xDI} = dm_{xc} - dm_{xg}$$

$$dm_{yUI} = dm_{yDI} = dm_{yc} - dm_{yg}$$

$$dm_{xUII} = dm_{xDII} = dm_{xc} - dm_{yg}$$

$$dm_{yUII} = dm_{yDII} = dm_{yc} - dm_{xg}$$

$$dm_{xUIII} = dm_{xDIII} = dm_{xc} - dm_{xg}$$

$$dm_{yUIII} = dm_{yDIII} = dm_{yc} - dm_{yg}$$

$$dm_{xUIV} = dm_{xDIV} = dm_{xc} - dm_{yg}$$

$$dm_{yUIV} = dm_{yDIV} = dm_{yc} - dm_{xg}$$

From suitable combinations, the differences $dm_{xc} - dm_{yc}$ and $dm_{xg} - dm_{xg}$

can be computed but absolute values of the scale corrections (or errors) can, of course, not be computed unless the grid or the comparator can be regarded free from scale errors.

From the corrections of the U-positions, the following expressions, for instance, are found:

$$dm_{xc} - dm_{yc} = \frac{dm_{xUI} - dm_{yUI} + dm_{xUII} - dm_{yUII} + dm_{xUIII} - dm_{yUIII} + dm_{xUIV} - dm_{yUIV}}{4}$$

$$dm_{xg} - dm_{yg} = \frac{-(dm_{xUI} - dm_{yUI}) + dm_{xUII} - dm_{yUII} - (dm_{xUIII} - dm_{yUIII}) + dm_{xUIV} - dm_{yUIV}}{4}$$

Since the scale differences are expressed in terms of quantities which are direct functions of measured data, the precision or the accuracy of the differences can be determined according to well-known procedures. Applied to the results of the computations as shown in Table XVI, the following values are found from the expressions for the scale differences:

$$dm_{xc} - dm_{yc} = 0.000020 \text{ and}$$

$$dm_{xg} - dm_{yg} = 0.000040$$

These data agree exactly with the introduced errors above.

b. <u>The Angular Errors</u> $d\beta$. Distinction can be made between a determination of the actual angles from the U- <u>or</u> D-positions or from the U- <u>and</u> D-positions. For the former case, we have from the basic differential formulas for the U-positions:

$$d\beta_{UI} = d\beta_g - d\beta_c$$

In the position UII, the error $d\beta_g$ changes its sign and, consequently,

$$d\beta_{UII} = -d\beta_g - d\beta_c$$

Further, $$d\beta_{UIII} = d\beta_g - d\beta_c$$

and $$d\beta_{UIV} = -d\beta_g - d\beta_c$$

After suitable combinations the following is found:

$$d\beta_c = - \frac{d\beta_{UI} + d\beta_{UII} + d\beta_{UIII} + d\beta_{UIV}}{4}$$

$$d\beta_g = \frac{d\beta_{UI} - d\beta_{UII} + d\beta_{UIII} - d\beta_{UIV}}{4}$$

In the D-position, the first expression is identical but the second one changes the sign since the grid is located upside down. Hence, the following is found:

$$d\beta_g = \frac{d\beta_{DII} - d\beta_{DI} + d\beta_{DIV} - d\beta_{DIII}}{4}$$

The angles $d\beta_c$ and $d\beta_g$ can also be directly found from a comparison between the corresponding U- and D-positions.

We have

$$d\beta_U = d\beta_g - d\beta_c$$

and $$d\beta_D = -d\beta_g - d\beta_c$$

Hence

$$d\beta_c = - \frac{d\beta_U + d\beta_D}{2}$$

$$d\beta_g = \frac{d\beta_U - d\beta_D}{2}$$

For the combinations of all adjustments in the U- and D-positions, we find:

$$d\beta_c = -\frac{d\beta_{UI} + d\beta_{DI} + d\beta_{UII} + d\beta_{DII} + d\beta_{UIII} + d\beta_{DIII} + d\beta_{UIV} + d\beta_{DIV}}{8}$$

$$d\beta_g = \frac{d\beta_{UI} - d\beta_{DI} - (d\beta_{UII} - d\beta_{DII}) + d\beta_{UIII} - d\beta_{DIII} - (d\beta_{UIV} - d\beta_{DIV})}{8}$$

From the laws of error propagation, the accuracy or the precision of the determined angles can be found.

If the data from Table XVI are substituted into the formulas derived, the errors of the angles are obtained which agree exactly with the introduced errors.

It should be emphasized that it would be sufficient to measure the nine points of the grid in two positions only, provided that a rotation through a right angle is made between the two positions. The quality of the determination of the regular errors will, however, be dependent upon the number of positions used. This can easily be shown in applying laws of error propagation to the expressions for the angles $d\beta_c$ and $d\beta_g$ above.

If two positions are used, the weight numbers are found from the following expressions:

$$d\beta_c = -\frac{d\beta_{UI} + d\beta_{UII}}{2}$$

$$d\beta_g = \frac{d\beta_{UI} - d\beta_{UII}}{2}$$

Since the two sets of measurements are regarded to be independent, the weight numbers are obtained as follows:

$$Q_{\beta c\beta c} = Q_{\beta g\beta g} = s_o \frac{Q_{\beta\beta}}{2}$$

and the standard error, after substituting Q from equation (50) or form 3:

$$s\beta_{c_2} = s\beta_{g_2} = s_o \frac{\sqrt{6}}{12a}$$

From four positions, we find similarly

$$s\beta_{c_4} = s\beta_{g_4} = s_o \frac{\sqrt{3}}{12a}$$

and eight positions

$$s\beta_{c_8} = s\beta_{g_8} = s_o \frac{\sqrt{6}}{24a}$$

Next, some examples of test measurements in an ordinary grid with a single image comparator will be shown.

10. Summary of Test Measurements in a Grid with a Single Image Comparator. A grid was measured in all eight positions in a comparator by a rather inexperienced operator. Nine points were used according to the derivation in point II3 above and in some experiments also 25 points according to Fig. 6 were measured (in one position only, however). All computations were performed with the forms 3 through 7 (Appendix). The summary of the results of different combinations of the measurements and computations will be shown.

a. Grid Positions U. First, the grid or the comparator is assumed to be errorless. The corrections dm_x, dm_y, and $d\beta$; their standard errors; and the standard error of unit weight all according to form 3 are shown for the four positions in Table XVII.

Table XVII. Regular Errors and Their Accuracy; All Data (Except the s_o) Given in Units of 10^{-6} mm

Position	dm_x	s_{m_x}	dm_y	s_{m_y}	$d\beta$	$s\beta$	s_o (microns)
UI	-27	11	+12	11	-3	15	2.6
UIII	-15	13	+15	13	-1	18	3.1
Average	-21		+14		-2		2.8
UII	+7	13	-15	13	-1	18	3.2
UIV	+27	10	+5	10	-1	14	2.4
Average	+17		-5		-1		2.8

For a judgment of the variations of the standard errors of unit weight and of the corrections, the confidence limits will be used on the level 5 per cent. For the standard error of unit weight, we find the limits $0.9\ s_o - 2.0\ s_o$

The found variations of s_o can, therefore, be regarded as normal. The confidence limits of the corrections are on the 5 per cent level (t-distribution) $\pm$ 2.2 s .

With respect to these confidence limits, only the corrections dm_x from the positions UI and UIV become significant. The choice of confidence level is, of course, somewhat arbitrary. In order to be sure that regular errors are adjusted as carefully as possible in the instrument(provided, of course, that the grid can be regarded as errorless), it seems advisable to use the standard error itself as the limit. In this case, where the number of redundant measurements (degrees of freedom) are 12, these confidence limits would correspond to a level of about 30 per cent.

From the formulas under IV9, the regular errors of the grid and the comparator can be distinguished. After substitution of the data from Table XVII, we find:

$$dm_{xc} - dm_{yc} = -0.000006$$

$$dm_{xg} - dm_{yg} = 0.000028$$

$$d\beta_c = 0.000002$$

$$d\beta_g = 0.000000$$

There is, consequently, a larger scale difference in the grid than in the comparator. The angular errors are very small. The comparator is evidently well adjusted. The standard errors of unit weight are 2.8 microns as an average. This is a combination of the errors of the grid and the comparator and includes also the errors of the operator.

In order to eliminate the errors of the grid in the determination of the standard error of unit weight, the four sets of measurements UI through UIV were mutually treated. The results of the measurements in position UI were used as given data, and the computations were made according to form 3(Appendix). The results concerning the basic quality should be denoted standard deviation since it is a question of repeated measurements of unknown data.

The following results were obtained:

UI - UII	s =	1.8 microns
UI - UIII	s =	1.7 microns
UI - UIV	s =	2.0 microns
Average:		1.8 microns

Since this is the standard deviation of a difference between two series of measurements, which can be assumed to be mutually equal, the standard deviation of one of the series can be obtained from dividing 1.8 by $\sqrt{2}$. This gives the value 1.3 microns, which is slightly greater than the standard deviation from repeated settings only.

b. Grid Positions D. The measurements in the positions D were treated in the same way as shown above for the positions U. The results concerning the regular errors of the grid and of the comparator agree well, and the differences which were found are, of course, due to the irregular errors.

The following data were obtained:

$$dm_{xc} - dm_{yc} = -0.000012$$

$$dm_{xg} - dm_{yg} = +0.000026$$

$$d\beta_c = 0.000007$$

$$d\beta_g = -0.000006$$

Also, in this case the largest scale difference was obtained for the grid.

The standard error of unit weight was found to be 3.3 microns as an average or somewhat larger than the corresponding value from the positions U. This may be explained by the fact that the measurements in the positions D were made through the glass of the grid and that small refraction errors may be caused by irregularities in the glass. A comparison between the results from the two locations U and D is shown in Table XVIII.

Table XVIII. Comparison Between the Results from U- and D-Locations of the Grid; All Data except Standard Errors of Unit Weight are in 10^{-6}

Position	dm_x	s_{m_x}	dm_y	s_{m_y}	$d\beta$	$s\beta$	s (microns)
UI	-27	11	12	11	-3	15	2.6
DI	-15	17	23	17	-20	23	4.0
UII	7	13	-15	13	-1	18	3.2
DII	-3	13	-20	13	-2	18	3.3
UIII	-15	13	15	13	-1	18	3.1
DIII	-33	13	2	13	-5	18	3.2
UIV	27	10	5	10	-1	14	2.4
DIV	9	11	-1	11	0	15	2.6

The computations with the aid of the forms 5 through 7 (Appendix) show very good agreement.

Also in the location D, the results of the measurements were transformed to the results of the series DI in order to find the standard deviation. The results were found as follows:

DI - DII	1.8 microns
DI - DIII	2.4 microns
DI - DIV	2.5 microns
Average:	2.2 microns

Consequently, a somewhat larger value was found from location D than from U. This may be explained by the influence of the glass of the grid.

c. Tests of the forms for the 25-point combination. Measurements and computations were also performed in 25 points according to form 4 (Appendix). The results were satisfactory, and the form seems to function well. The standard error of unit weight was found to be 3.4 microns. This value is slightly larger than the results from the measurements and computations as shown above. Some of the grid points in the 25-point combination were of considerably lower quality than the points used in the 9-point combination. This may explain the difference.

Further examples of the 25-point combination will be shown below in connection with the tests of some different modern stereocomparators.

11. Practical Determination of the Absolute Scale of a Comparator and the Basic Accuracy. Two different scales and comparators have been used. From Dr. Francis E. Washer, Refractometry Section, Metrology Division, National Bureau of Standards, some sets of observations for the determination of the absolute scale and accuracy of a comparator were obtained. These observations have been computed according to the principles shown above in Section III7. First, an adjustment was made of data which were averages of the measurements of two observers and of ten repeated settings. The results are shown in Tables XIX and XX and in Figs. 16 and 17. Further, the results of each individual observer were treated in a similar manner. The results of these computations are shown in Figs. 18 and 19.

a. The Actual Computations.

Table XIX. Measured Data and Residuals; Averages of Two Observers			
Scale Line (mm)	X (mm)	dx (microns)	v resid. (microns)
20	-160	-0.05	+0.59
30	-150	+0.03	+0.69
40	-140	1.25	-0.35
50	-130	1.24	-0.16
60	-120	1.50	-0.24
70	-110	1.06	+0.38
80	-100	1.57	+0.06
90	- 90	1.21	+0.60
100	- 80	1.17	+0.82
110	- 70	2.36	-0.19
120	- 60	2.55	-0.20
130	- 50	2.41	+0.13
140	- 40	2.76	-0.04
150	- 30	2.86	+0.04
160	- 20	3.52	-0.44
170	- 10	3.11	+0.15
180	0	3.82	-0.37
190	+ 10	4.35	-0.72
200	+ 20	4.31	-0.50
210	+ 30	4.62	-0.63
220	+ 40	5.12	-0.95
230	+ 50	4.14	+0.21
240	+ 60	4.39	+0.15
250	+ 70	5.17	-0.45
260	+ 80	5.56	-0.66
270	+ 90	5.42	-0.34
280	+100	6.12	-0.86
290	+110	5.97	-0.53
300	+120	5.37	+0.26
310	+130	5.30	+0.51
320	+140	5.09	+0.81
330	+150	4.82	+1.35
340	+160	5.56	+0.79
5940	0	113.68	-0.09

X = the coordinates along the scale, referred to the point of gravity as origin.

dx = the discrepancies between the comparator and the glass scale, defined as errors i.e. measured value minus given value or comparator coordinates minus glass scale coordinates.

dx was determined as averages of 10 repeated settings in each point.

v = the residuals after the adjustment, computed for a check of the sum of the squares of the residuals and for the histogram, used in the normal distribution test.

From the Table: $n = 33$

$[dx] = 113.68$

$[Xdx] = 5440.10$

$[XX] = 299200$

$[dxdx] = 500.6716$

$$dx_o = \frac{113.68}{33} = -3.4448 \text{ microns}$$

$$dm_x = -\frac{5440.10}{299200} = -0.018182 \text{ micron/mm}$$

$$[vv] = 500.6716 - 391.6103 - 98.9127 = 10.1486$$

$$s_o = \sqrt{\frac{10.1486}{31}} = 0.57 \text{ micron}$$

The residuals v were computed from the formula (130) after substitution of the parameters from the adjustment.

Table XX. Test of Normal Distribution of Residuals After Adjustment.

Class means t 0.1 micron	Number of values f	ft	ft^2	Class limits b	Stand. class limits $(b-\bar{x}):s$	Probability p	np	f-np	$\frac{(f-np)^2}{np}$
14	1	14	196	13	2.30	0.0107	0.3	0.2	0.05
12	0	0	0	11	1.94	0.0155	0.5		
10	0	0	0	9	1.59	0.0297	1.0	-1.0	1.00
8	3.5	28	224	7	1.24	0.0516	1.8	1.7	1.61
6	3	18	108	5	0.89	0.0792	2.6	0.4	0.06
4	2	8	32	3	0.53	0.1114	3.7	-1.7	0.73
2	4.5	9	18	1	0.18	0.1305	4.3	0.2	0.01
0	3	0	0	-1	-0.18	0.1428	4.7	-1.7	0.61
-2	4.5	-9	18	-3	-0.53	0.1305	4.3	0.2	0.01
-4	6.5	-26	104	-5	-0.89	0.1114	3.7	2.8	2.12
-6	2	-12	72	-7	-1.24	0.0792	2.6	-0.6	0.14
-8	1.5	-12	96	-9	-1.59	0.0516	1.8	-0.3	0.05
-10	1.5	-15	150	-11	-1.94	0.0297	1.0	0.5	0.25
-12	0	0	0	-13	-2.30	0.0155	0.5	-0.5	0.50
-14	0	0	0						7.19
	n=33	+3	1018						

$\bar{x} = \frac{[ft]}{n} = 0$

$s = \sqrt{\frac{1018}{32}} = 5.7$

Degrees of freedom = 13-3 = 10

$Chi^2_{95} = 3.940$

$Chi^2_{5} = 18.307$

The value 7.19, consequently, means that the hypothesis that a normal distribution is present can be accepted on the 5 per cent level.

A histogram of the residuals and the corresponding normal distribution curve is shown in Fig. 17.

b. Summary and Discussion of the Adjustment Procedure. Through the adjustment computation, the two systematic or regular errors dx_o and dm_x have been determined according to the method of least squares. The sum of the squares of the residuals [vv] has been used for calculation of the standard error of unit weight s_o which can be regarded as the statistical expression for the irregular errors which remain after the correction for the two regular errors dx_o and dm_x.

The obtained value of the standard error of unit weight s_o = 0.57 micron is an expression for the errors of the glass scale, the comparator, and the operator. Doubtless, there are regular errors of different kinds, for instance, caused by temperature variations or of periodic nature in the scales. A certain distinction might be possible after examination of the residuals which are shown in Table XIX, the right column. In Fig. 16, the residuals are shown graphically. It seems possible to determine a kind of sine-curve, which might reduce the sum of the squares of the residuals. Before doing this, however, it would be suitable to make another series of tests measurements and computations similar to those above in order to see if the residuals become distributed in approximately the same manner as shown here. If this is the case, a determination of the sine-curve numerically and according to the method of least squares should be performed.

For the judgment if the standard error of unit weight can be regarded as an expression for irregular errors, a normal distribution test of the residuals is suitable. Such a test is shown in Table XX and Fig. 17. The results of the test show that the residuals are normally distributed according to the chi^2 criterion on the 5 per cent level. The number of residuals (33) is, however, too small for a completely satisfactory test. The results confirm the central limit theorem, however, concerning the normal distribution.

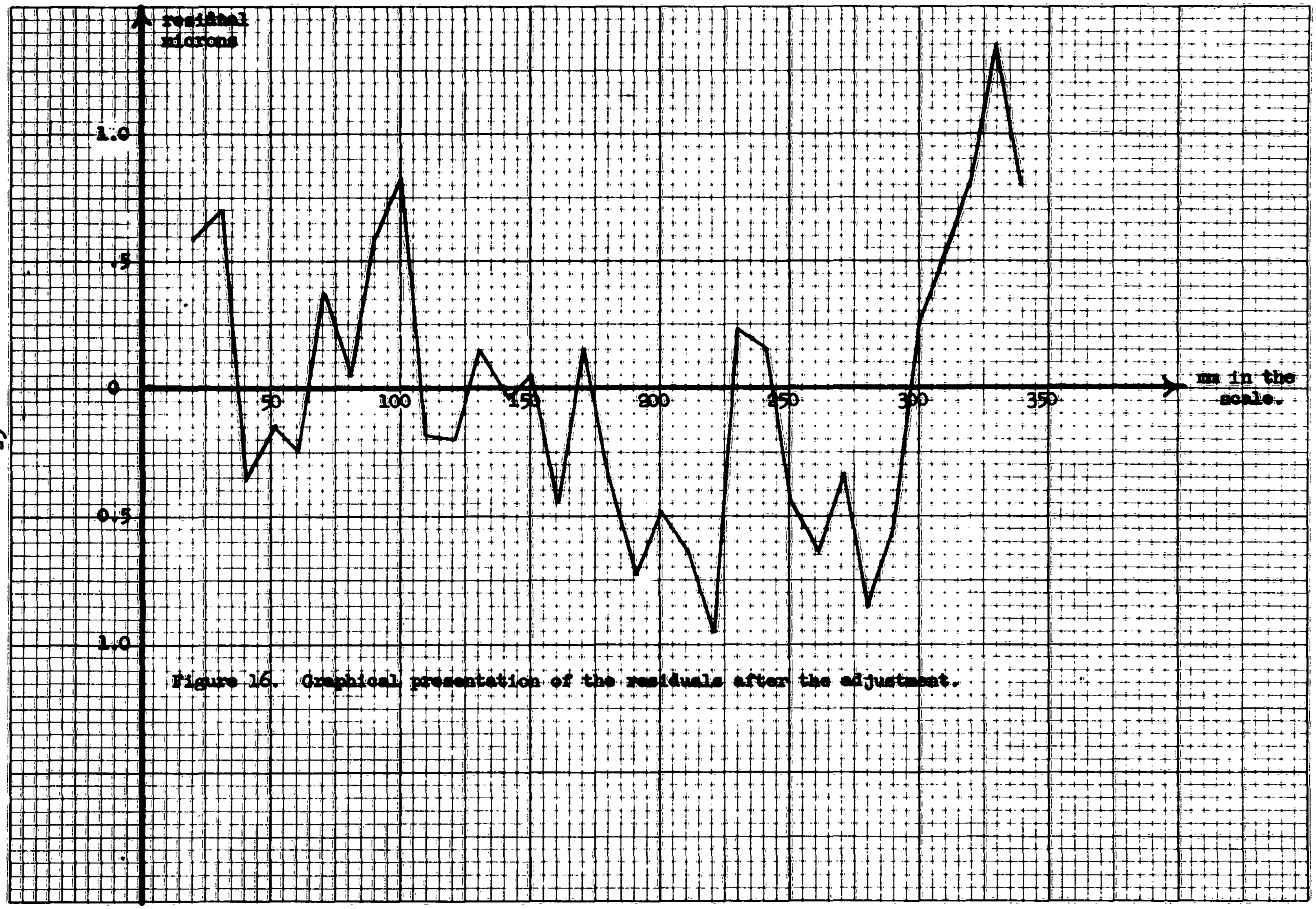


Figure 16. Graphical presentation of the residuals after the adjustment.

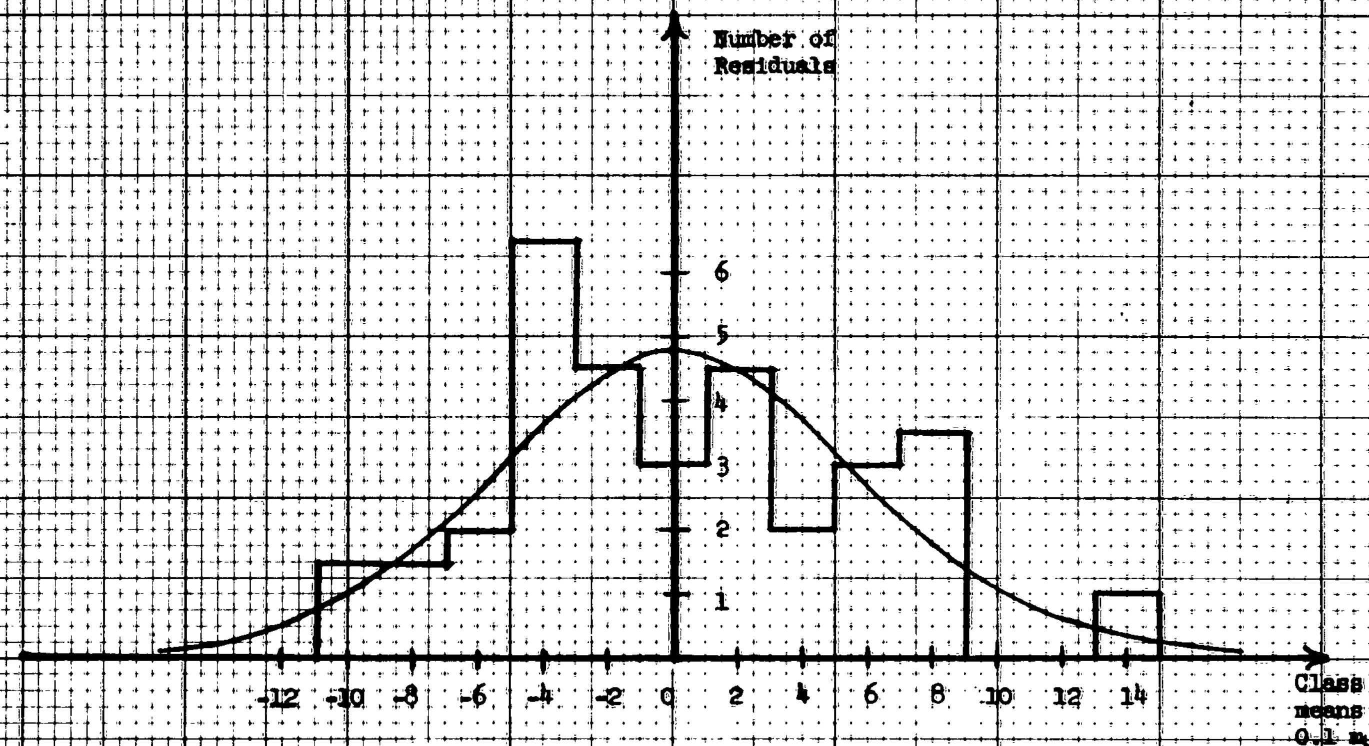


Figure 17. Histogram of the residuals and the corresponding normal distribution curve.

Assuming the glass scale coordinates to be of approximately the same quality as the coordinates of the comparator, the standard error of each of these sets of coordinates can be computed from the found value $s_o = 0.57$ micron by dividing this by $\sqrt{2}$ or 1.414. We then find the value 0.4 micron which can be regarded as a good estimation of the accuracy of the coordinates of the glass scale and of the coordinates of the comparator. In the mentioned value, the errors of the operator are included. Further, it must be emphasized that each coordinate was determined as the average of ten settings and that the standard error of unit weight refers to this method of measurements.

The results are of great practical interest since they show the accuracy that can be obtained in using the actual scales and instruments for absolute scale determination.

The observed series from the two operators, A and B, were treated separately. The observed data were averages of five settings each. The procedure was identical with what has been shown above. From the results shown in Fig. 18, 18a and 18b, the conclusion can be found that there may be a systematic error of periodic nature in the residuals. The agreement between the residuals from the two observers is striking and indicates a sine-shaped fluctuation, which also might be adjusted. The residuals are, in both cases, normally well distributed on the 5 per cent level and the standard errors of unit weight are about 0.5 and 0.7 micron, respectively. Assuming the accuracy of the comparator and of the glass scale to be approximately the same, the standard error of unit weight of the comparator becomes of the order of magnitude 0.4 micron. This is a very good accuracy, which evidently can be obtained.

Another set of measurements in a comparator of a glass scale was obtained from Dr. Markowitz, U. S. Naval Observatory, and has been computed in a similar way as was shown above. A brief summary of the computations and the results will be given here.

A 17-cm scale on glass was measured in a linear comparator. From repeated settings in each line the precision of the measurements could be determined as a standard deviation of one measurement and of the average of each series of settings.

The results of the measurements of 17 lines are shown in Table XXI. Since the task of the measurements primarily was to test the ruling of the glass scale, the results of the measurements in the comparator are regarded as given data and the nominal values of the glass scale as measured data. The errors e are defined as dx = measured value - given value. In the table also the coordinates X of the lines are shown, as determined from the point of gravity of the scale.

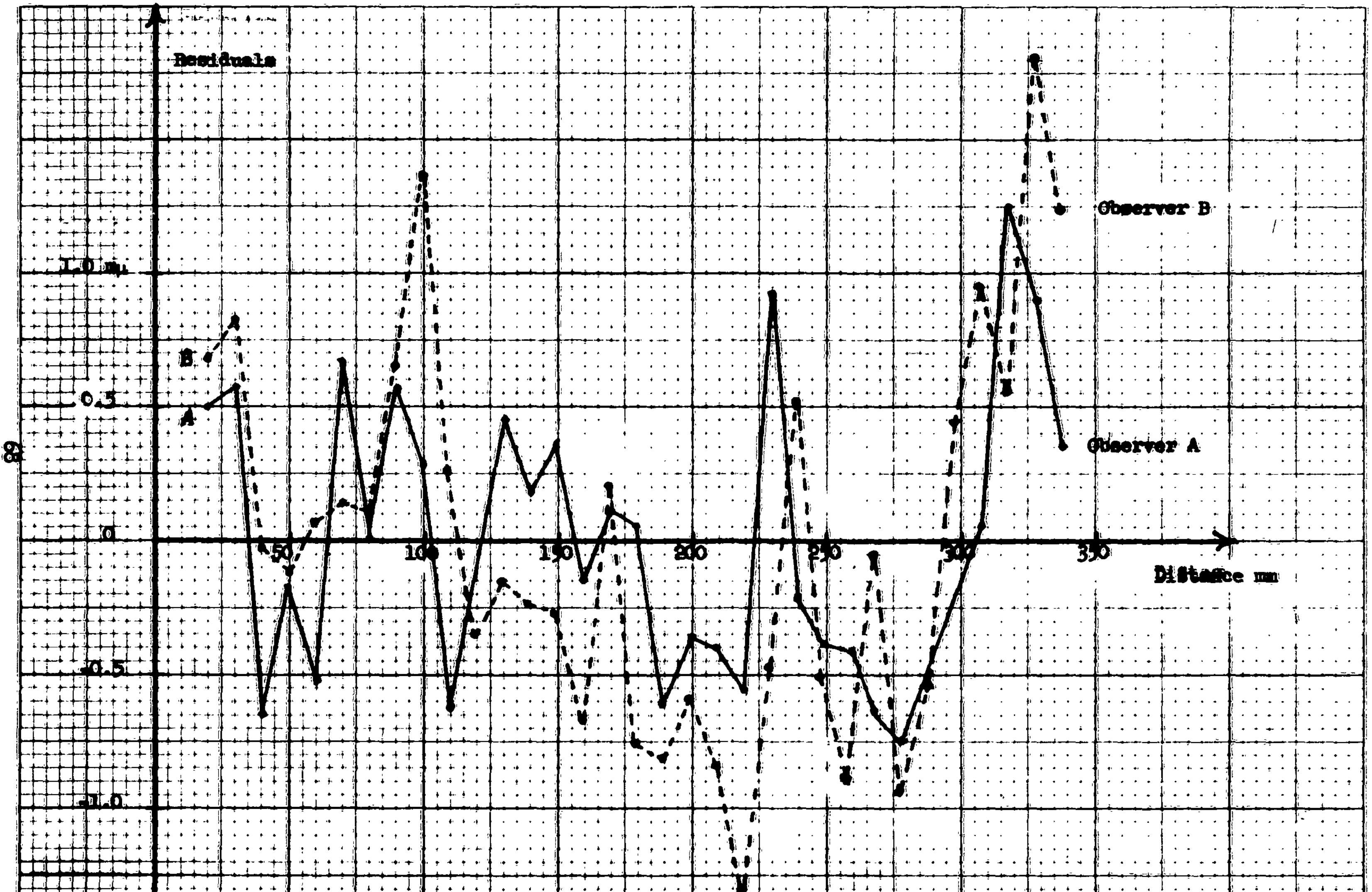


Figure 18. Distribution of the residuals after the adjustment of the comparator measurements; Two observers.

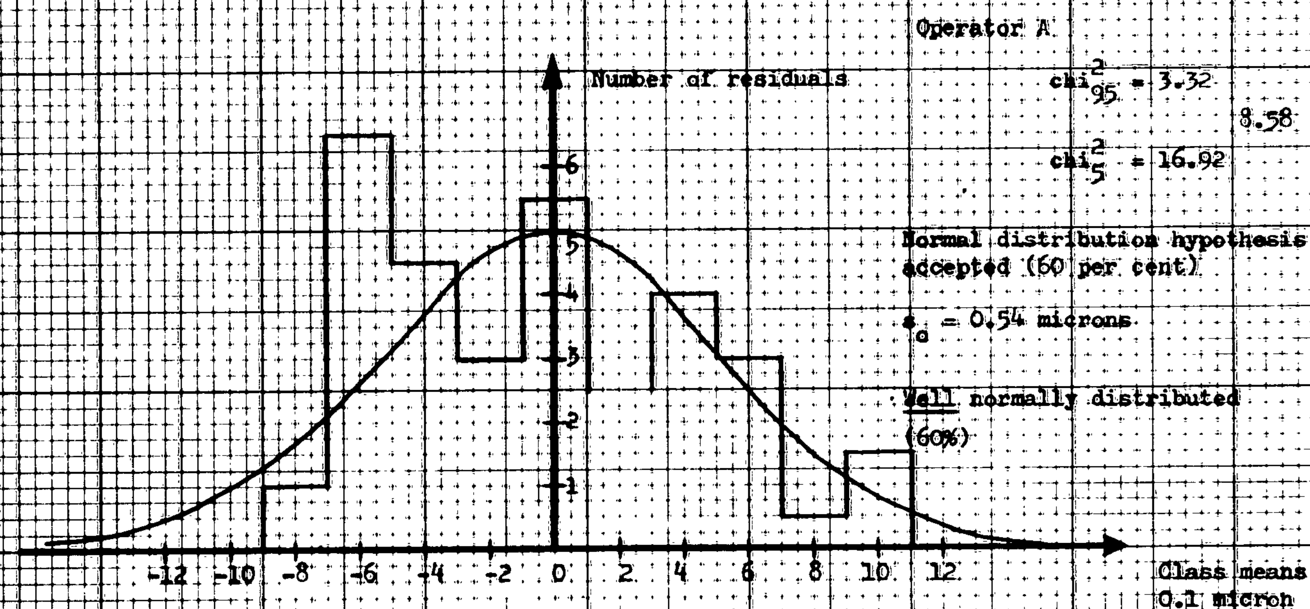


Figure 18a. Histogram and normal distribution curve of the residuals after adjustment of comparator measurements in glass-scales. National Bureau of Standards. Operator A.

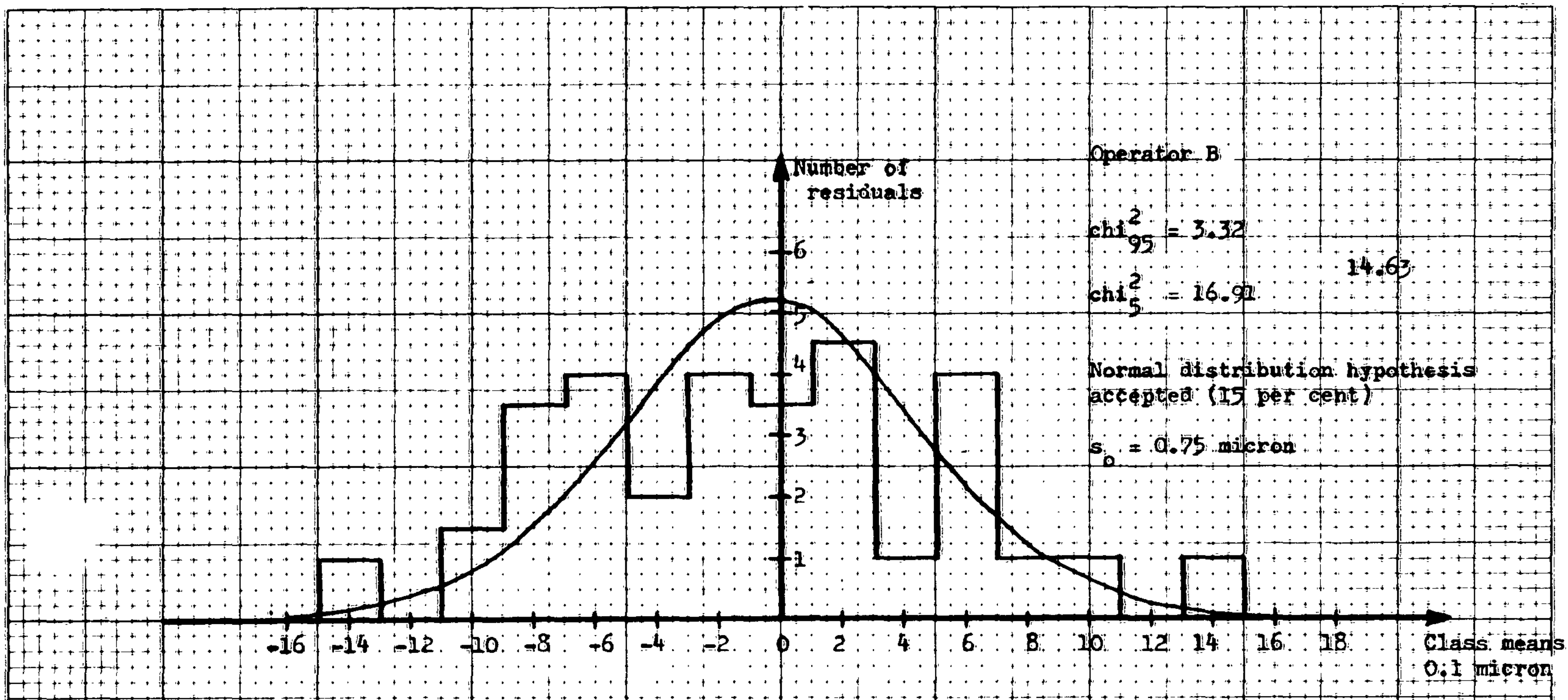


Figure 18b. Histogram and normal distribution curve of the residuals after adjustment of comparator measurements in glass-scales. National Bureau of Standards. Operator B.

Table XXI. Measured and Given Data. Errors			
Measured Scale Line mm	Given Values mm	Error dx microns	Coord. X mm
0.000	0.000	0	-85
10.000	9.9996	+0.4	-75
20.000	19.9999	+0.1	-65
30.000	30.0001	-0.1	-55
40.000	40.0001	-0.1	-45
50.000	50.0000	0.0	-35
60.000	60.0001	-0.1	-25
70.000	70.0005	-0.5	-15
80.000	80.0003	-0.3	- 5
90.000	90.0006	-0.6	+ 5
100.000	100.0006	-0.6	+15
110.000	110.0012	-1.2	+25
120.000	120.0011	-1.1	+35
130.000	130.0012	-1.2	+45
140.000	140.0015	-1.5	+55
150.000	150.0016	-1.6	+65
160.000	160.0017	-1.7	+75
170.000	170.0019	-1.9	+85

The following computations are made directly from the table.

$n = 18 \quad [dx] = -12 \qquad dx_o = +0.67$ microns

$[Xdx] = -6250 \quad [XX] = 48450 \qquad dm_x = 0.012900$ microns/mm

$[dxdx] = 16.66$

$$[vv] = 16.66 - \frac{144}{18} - \frac{625^2}{48450} = 16.66 - 16 = 0.66$$

$$s_o = \sqrt{\frac{0.66}{16}} = \sqrt{0.04} = 0.2 \text{ microns}$$

$$s_{s_o} = \frac{s_o}{\sqrt{32}} = 0.04 \text{ microns}$$

Confidence limits: t-distribution, confidence level 5 per cent:
$\pm 2.1\, s_o = \pm 0.4$ microns.

Confidence level 1 per cent:
$\pm 2.9\, s_o = \pm 0.6$ microns.

The residuals are shown in Figure 19.

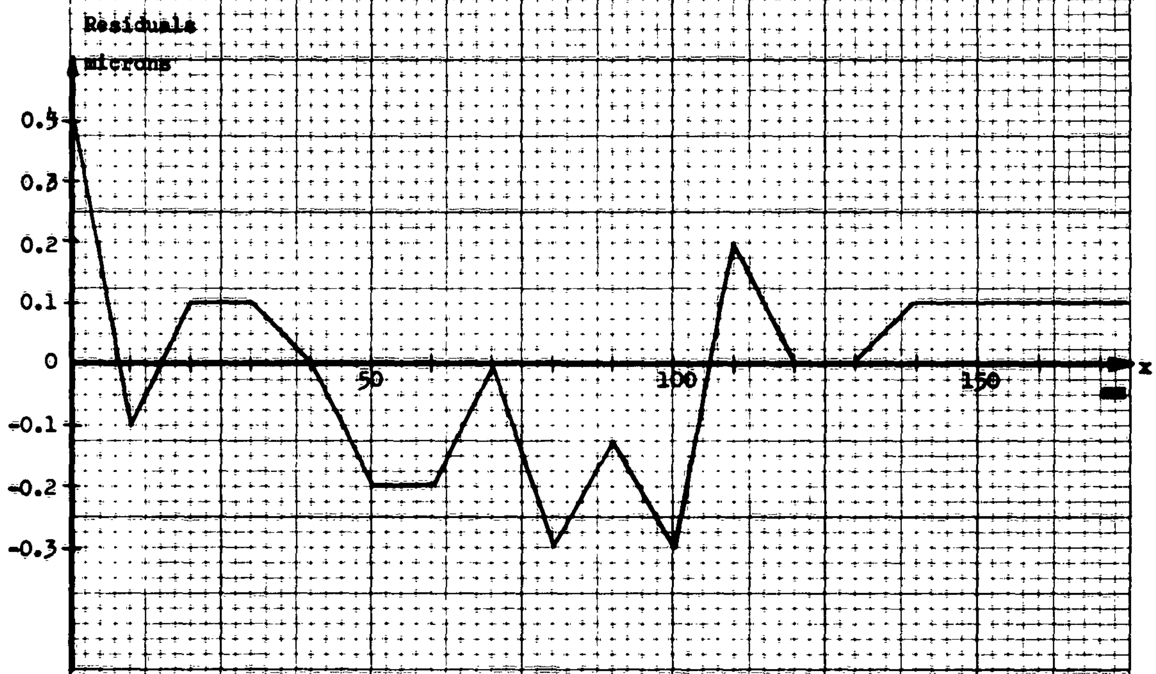


Figure 19. Residuals along the glass scale after the adjustment.

Summary: The accuracy of the measurements is very high and indicates that the glass scale and the comparator are of very high quality. The glass scale can consequently be used for the determination of absolute scales of other comparators. Lacking orthogonality can be checked with the aid of an ordinary grid, which is measured in at least two, but preferably 4 different positions after rotation through a right angle between each position.

12. Practical Tests of Some Different Stereocomparators. Primarily, in order to determine the standard error of unit weight of image coordinate measurements for the establishment of tolerances, etc., but also for practical test of the derived formula systems, a glass grid of high and known accuracy has been measured in some different stereocomparators. The glass grid used was specially ordered from the Wild Company in Switzerland and was delivered with the coordinates of all grid intersections. The grid division is in units of 10 x 10 mm.

Check measurements of the grid were made at the Division of Photogrammetry of the Institute of Technology in Stockholm, Sweden. From the completely independent measurements and computations (adjustments of coordinate transformations), the standard error of the grid coordinates as delivered by the Wild Company was found to be about 1 microm. This is a very high accuracy, and the grid can be regarded to be of highest possible quality. A description of the check procedure was given in the paper; Lycken, L. E., Test Measurements of Grids, Int. Archives of Photogrammetry, Vol. XII, Stockholm 1956, Comm. II.

The grid was approximately adjusted in one of the image holders of the comparator, and the coordinates of 25 points were first measured and recorded. Electronic recording devices were used in all cases. In one of the points, a great number of repeated settings were made in order to determine the precision of the coordinate measurements. Further, in some cases, a number of additional points were measured for statistical tests of the residuals after corrections with respect to the adjustment in the 25 points.

A rather detailed description of one of the experiments performed will be given while only the final results of the others are presented.

a. Stereocomparator I.

(1) The Precision of the Coordinate Measurements. In an arbitrary point of the grid, 22 repeated settings were made and the corresponding x- and y-coordinates were automatically recorded. The averages were computed, and the deviations between the averages and the individual observations were determined. The standard deviation of one measurement was then found to be 1 micron for x and y. The precision can be increased through using repeated settings in

each point and determining the average since the standard deviation of the average decreases with the square root of the number of observations. It is evidently an economical problem how many observations should be made. At least two observations and preferably three might be advisable in case of, for instance, delivery tests of an instrument. In the actual case, only one setting was made due to the limited time. It might be noted at this point that the importance of the standard deviation sometimes is overestimated in the determination of the accuracy of an instrument. Theoretically, it would be possible to increase the precision infinitely by increasing the number of settings but the accuracy cannot be similarly increased.

(2) The Accuracy of the Comparator. From the measurements in the 25 points, an adjustment was made with the aid of forms 1 and 4. The comparator coordinates were, in other words, transformed to the system of the given grid coordinates and the elements of the transformation were determined under the conditions that the sum of the squares of the residuals is a minimum. The results of the computations:

$$dx_o = 2.52 \text{ microns}$$

$$dy_o = 1.72 \text{ microns}$$

$$dm_x = 0.0000192$$

$$dm_y = 0.0000068$$

$$d\alpha = -0.0005484 \text{ radians} = -3^c.49^{cc}$$

$$d\beta = -0.0000056 \text{ radians} = -3.^{cc}6$$

$$[vv] = 76 \text{ microns}^2$$

$$s_o = 1.3 \text{ microns}$$

There is a small scale error found. In the x-direction, the quantity dm_x means 4.8 microns for 240 mm, and in the y-direction the corresponding dm_y means 1.3 microns. Consequently, the affine deformation is to be taken into account if standard errors of about 1 micron in magnitude are to be desired in the results of coordinate measurements. The lacking orthogonality was found to be very small and can be neglected. The standard error of unit weight (1.3 microns) indicates a very good quality of the instrument. It should be noted that the grid can hardly be regarded as errorless in comparison with the standard error of unit weight obtained. If the standard error of the grid coordinates, 1 micron, is quadratically subtracted from

the standard error of unit weight, there remains a standard error of the comparator of about 1 micron also. This means that the standard error is of the same order of magnitude as the standard deviation of the coordinate measurements and that the instrument, consequently, is as good as possible after corrections for the determined regular errors, primarily the scale errors. It also proves that the accuracy of a comparator really can be so high that the standard error of unit weight becomes about 1 micron.

(3) Determination and Statistical Tests of the Residuals after the Adjustment. It is most important to be able to check whether or not the residual errors or corrections after an adjustment are normally distributed. For such purposes, the individual residuals must be computed. The residuals are obtained from the working correction equations after substitution of the parameters from the normal equations.

In this case, the residuals were found from the following expressions:

$$v_x = 2.5 - 0.000019x + 0.000554y - dx$$

$$v_y = 1.7 - 0.000007y - 0.000548x - dy$$

Hence, after substitution of the x- and y-coordinates and the corresponding coordinate errors, the individual residuals were found. For the statistical test, the residuals were divided into class intervals of 0.8 microns, viz., -4.4 to -3.6, -3.6 to -2.8, -2.8 to -2.0, -2.0 to -1.2, -1.2 to -0.4, -0.4 to +0.4, +0.4 to +1.2, +1.2 to +2.0, +2.0 to 2.8, +2.8 to +3.6, +3.6 to +4.4.

Residuals which were identical with the class limits were distributed with $\frac{1}{2}$ to each of the actual classes. The test procedure is shown in Table XXII.

Table XXII. Normal Distribution Test

Class Means t	Residuals f	ft	ft^2	Class limits b	Standardized class limits $\frac{b-\bar{x}}{s}$	p	np	$f-np$	$\frac{(f-np)^2}{np}$
-4.0	1	-4.0	16.00	-3.6	-2.77	0.0028			
-3.2	0.5	-1.6	5.12	-2.8	-2.15	0.0130	3.0	0.0	0.00
-2.4	1.5	-3.6	8.64	-2.0	-1.54	0.0158 0.0460			
-1.6	1.5	-2.4	3.84	-1.2	-0.92	0.1170	5.8	-4.3	3.19
-0.8	13	-10.4	8.32	-0.4	-0.31	0.1995	10.0	+3.0	0.90
0.0	15.5	0	0	+0.4	+0.31	0.2434	12.2	+3.3	0.89
+0.8	8.5	6.8	5.44	+1.2	+0.92	0.1995	10.0	-1.5	0.22
+1.6	5.5	8.8	14.08	+2.0	+1.54	0.1170	5.8	-0.3	0.02
+2.4	2	4.8	11.52	+2.8	+2.15	0.0460			
+3.2	1	3.2	10.24	+3.6	+2.77	0.0130	3.0	0.0	0.00
+4.0	0	0	0			0.0028			
$n = 50.0$		1.6	83.20				49.8		5.22

$$s = \sqrt{\frac{83.20}{50}} = 1.3 \text{ microns}$$

$$\bar{x} = \frac{1.6}{50} = 0$$

Degrees of freedom $= 7 - 3 = 4$

$$\text{chi}^2_{95} = 0.71 \qquad \text{chi}^2_{5} = 9.49$$

The probability p has been taken from a table of the normal distribution function.

The result 5.22 is between 0.71 and 9.49. The residuals are normally distributed on the 5 per cent level. 5.22 corresponds approximately to a level of about 60 per cent.

A histogram and the corresponding normal distribution curve are shown in Fig. 20.

(4) Measurements and Tests of Additional Grid Coordinates. After the measurements of grid coordinates in 25 regularly located points (Form 1, Appendix), 36 points were measured along the diagonals of the grid. The results of these measurements have been used as a check of the error propagation from the adjustment of the original measurements in the 25 points.

First, the measured grid coordinates were translated to the same system as the original 25 points before the adjustment. Corrections and residuals were then computed in the same manner as was shown above. An inspection of the residuals proved that a certain

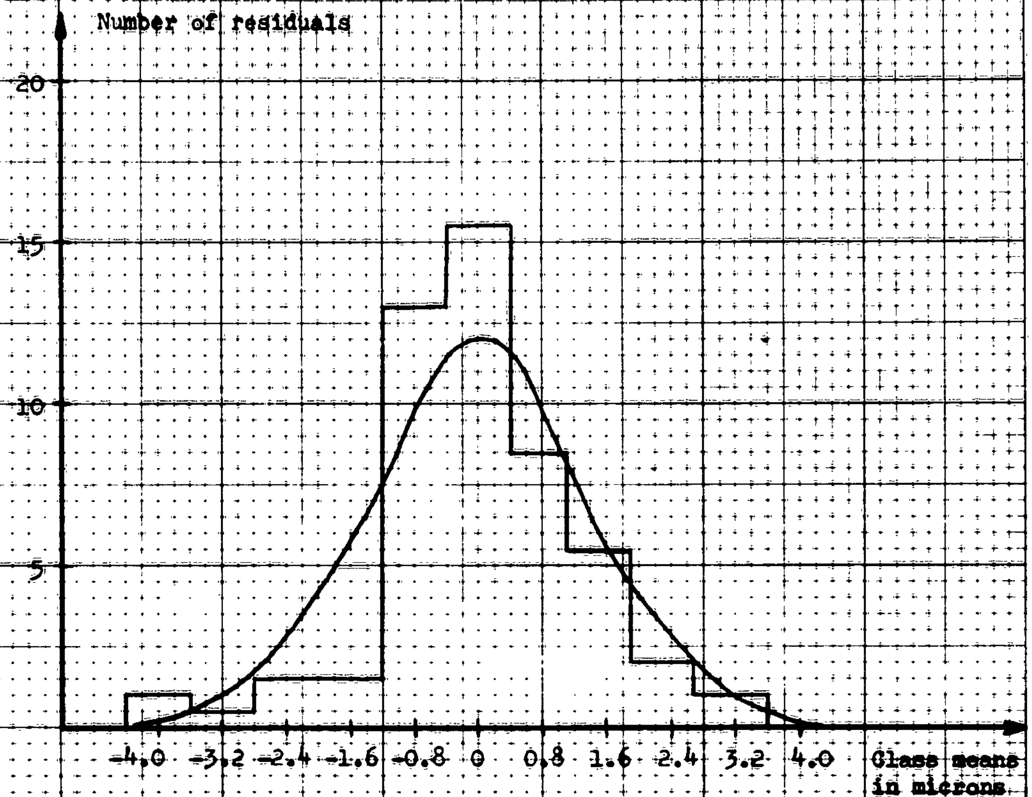


Figure 20. Histogram and normal distribution curve of residuals after adjustment of grid coordinate measurements in a stereocomparator.

constant error was present in the x-residuals. This may have been caused by a minor vibration of the instrument between the two series of measurements. The magnitude of the constant value of the residuals was 2.5 microns. This value was subtracted from all x-residuals. The y-residuals showed no similar tendency. The root mean square values of the residuals were found to be:

In $x = 1.8$ micron, and in $y = 1.9$ micron.

Next, it is of interest to test if these results agree theoretically with the error propagation from the basic measurements according to the laws of error propagation (See under section II4 above, expression (82)).

For $a = 50$ mm, we find the root mean square values of the standard errors $M_{s_x} = M_{s_y} = 1.06\ s_o = 1.4$ micron. The agreement with the practically determined root mean square values 1.8 and 1.9 micron is not very good. The question arises as to whether or not the differences are significant. If they are significant, the errors of the residuals in additional points may have been caused by other sources of errors than those which caused the standard error of unit weight. In such cases, the chi-square test can be used. According to well-known procedures, we compute the confidence limits of the standard error for two confidence levels 5 and 1 per cent and for 44 degrees of freedom. The number of observations is 50. We find the following confidence limits:

For 5-per cent confidence level:	$0.88\ s_o - 1.35\ s_o$ or
	1.2 - 1.9 micron
For 1-per cent confidence level:	$0.83\ s_o - 1.45\ s_o$ or
	1.2 - 2.0 micron

Consequently, the upper limits are reached for the 5-per cent level and nearly also for the 1-per cent level. The relation between theory and practice can, therefore, be accepted although with some hesitation. The explanation for the high root mean square values from the practical tests may be that the measurements were made in one long, nearly uninterrupted series, which lasted more than one hour. Therefore, the results of the measurements of the additional points may include errors due to tiredness.

The residuals in the additional points were also tested concerning the possible normal distribution. This test was performed in a similar manner as above (See Table XXIII).

Table XXIII. Residual Computations

Class Means t	Residuals f	ft	ft^2	Class limits b	Standardized class limits $(b-\bar{x}):s$	p	np	f-np	$\frac{(f-np)^2}{np}$
-4.8	2	-9.6	46.08	-4.4	-2.44	0.0073 0.0155 0.0366 0.0594	4.3	0.2	0.01
-4.0	0	0	0	-3.6	-2.00				
-3.2	2.5	-8.0	25.60	-2.8	-1.56				
-2.4	4	-0.6	23.04	-2.0	-1.11	0.0741	5.3	-1.3	0.32
-1.6	7.5	-12.0	19.20	-1.2	-0.67	0.1179	8.5	-1.0	0.12
-0.8	14	-11.2	8.96	-0.4	-0.22	0.1615	11.6	+2.4	0.91
0	13.5	0	0	+0.4	+0.22	0.1742	12.6	+0.9	0.06
+0.8	10.5	8.4	6.72	+1.2	+0.67	0.1615	11.6	-1.1	0.10
+1.6	9	14.4	23.04	+2.0	+1.11	0.1179	8.5	+0.5	0.03
+2.4	6	14.4	34.56	+2.8	+1.56	0.0741	5.3	+0.7	0.09
+3.2	2	6.4	20.48	+3.6	+2.00	0.0366 0.0288 0.0155 0.0073	4.3	-1.3	0.39
+4.0	1	4.0	16.00	+4.4	+2.44				
+4.8	0	0	0						
	72	1.2	223.68				72.0		2.03

$s = 1.8$ micron

$\bar{x} = 0$

Degrees of freedom = 9 - 3 = 6

$chi^2_{95} = 1.64$ $\quad$ $chi^2_5 = 12.59$

The histogram and normal distribution curve are shown in Fig. 21.

b. Stereocomparator II. In a similar way as has been described above in Section IV 12a, another stereocomparator has been tested by measurements in the grid. Only the 25 basic points were measured; however, but the computations were varied in order to give a more complete information on the distribution of the errors.

First, the adjustment of the measurements in all 25 points was performed and the following results were obtained.

$dx_o = 0.04$ microns $\qquad s_{x_o} = 0.18$ microns $= s_{y_o}$

$dy_o = -0.24$ microns $\qquad s_{m_x} = 0.0000047 = s_{m_y}$

$dm_x = 0.0000232 \qquad s\alpha = 3\overset{cc}{.}0$

$dm_y = -0.0000220 \qquad s\beta = 4\overset{cc}{.}3$

$d\alpha = 16\overset{cc}{.}0 \qquad s_o = 1.7$ microns

$d\beta = 1\overset{cc}{.}0$

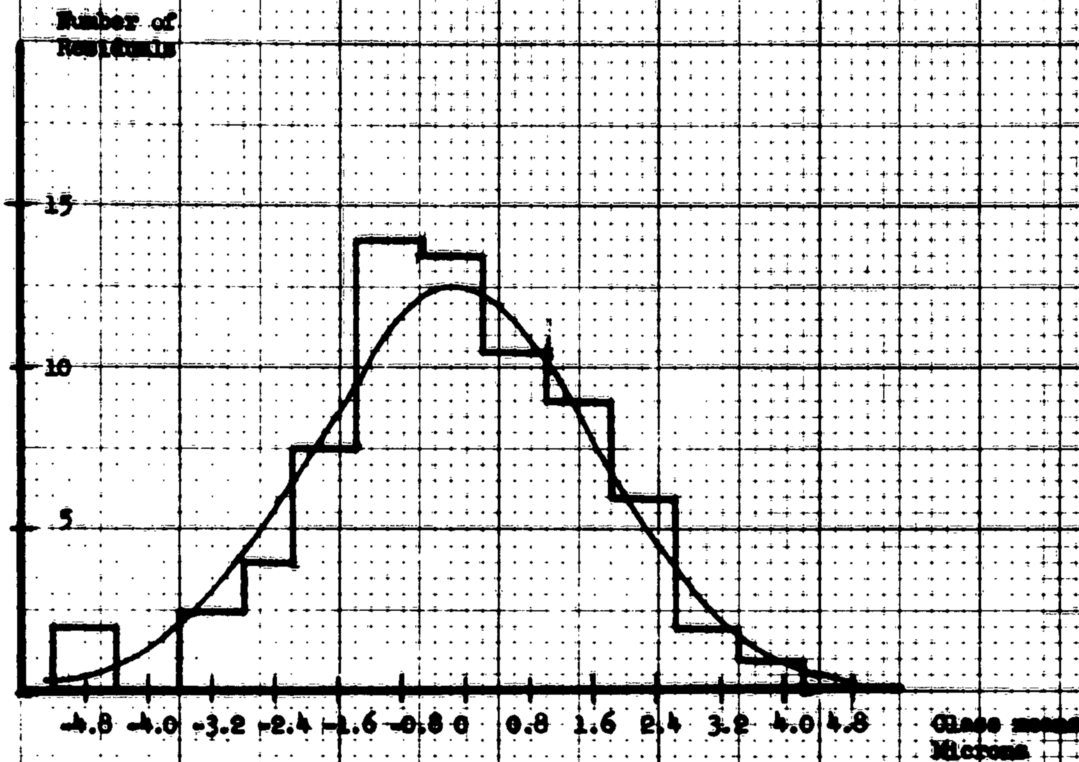


Figure 21. Histogram and normal distribution curve of residuals in additional points after adjustment of grid coordinate measurements in a stereocomparator.

After correction for the standard error of the grid is found $s_o = 1.4$ microns. The residuals were computed in all points and were tested concerning the statistical distribution. The normal distribution was found in both cases (x and y) (Fig. 22). Of course, the number of residuals is comparatively small for a test like this. For a delivery test, for instance, a great number of additional points should be measured and taken into account. A graphical presentation of the residuals is shown in Fig. 23.

The coordinate transformation and adjustment was also made with five points only. The points were located in the corners and in the center of the grid. After the transformation and adjustment, the residuals were computed and tested for normal distribution. The histogram of all residuals in x and y, except for those in the transformation points and the corresponding normal distribution curve are shown in Fig. 24,and the individual residuals are shown graphically in Fig. 25.

c. Stereocomparator III. It was found that the original grid could not be used in this comparator since the format of the plateholders was somewhat too small. Therefore, a contact print on glass of the original grid had to be used for the test measurements in this instrument. The contact print had been measured in one of the other comparators immediately after the measurements of the original grid in the same comparator. Therefore, the accuracy of the contact print could be estimated from these measurements.

In Comparator III, 25 points were measured in the contact print and these measurements were adjusted with respect to the measurements in the same points in the other comparator. The results of these computations indicated a standard error of unit weight of the image coordinate measurements in Comparator III of 1.6 microns; i.e., the same order of magnitude as was found in Comparators I and II.

V. SUMMARY OF THE PRACTICAL TESTS

The practical tests have proved that the developed formula systems work correctly and that they are simple to handle. The results of the practical measurements give information about the accuracy and precision which can be expected from instruments which are available today for image coordinate measurements. In most cases, regular (correctable) errors have been found in the instruments. In some cases, further adjustments of the instruments may be possible. It may well be discussed, however, how far such an adjustment should be made mechanically. An adjustment can evidently never be made free from errors, and the last corrections must be made numerically. In such a situation, the order of magnitude of the

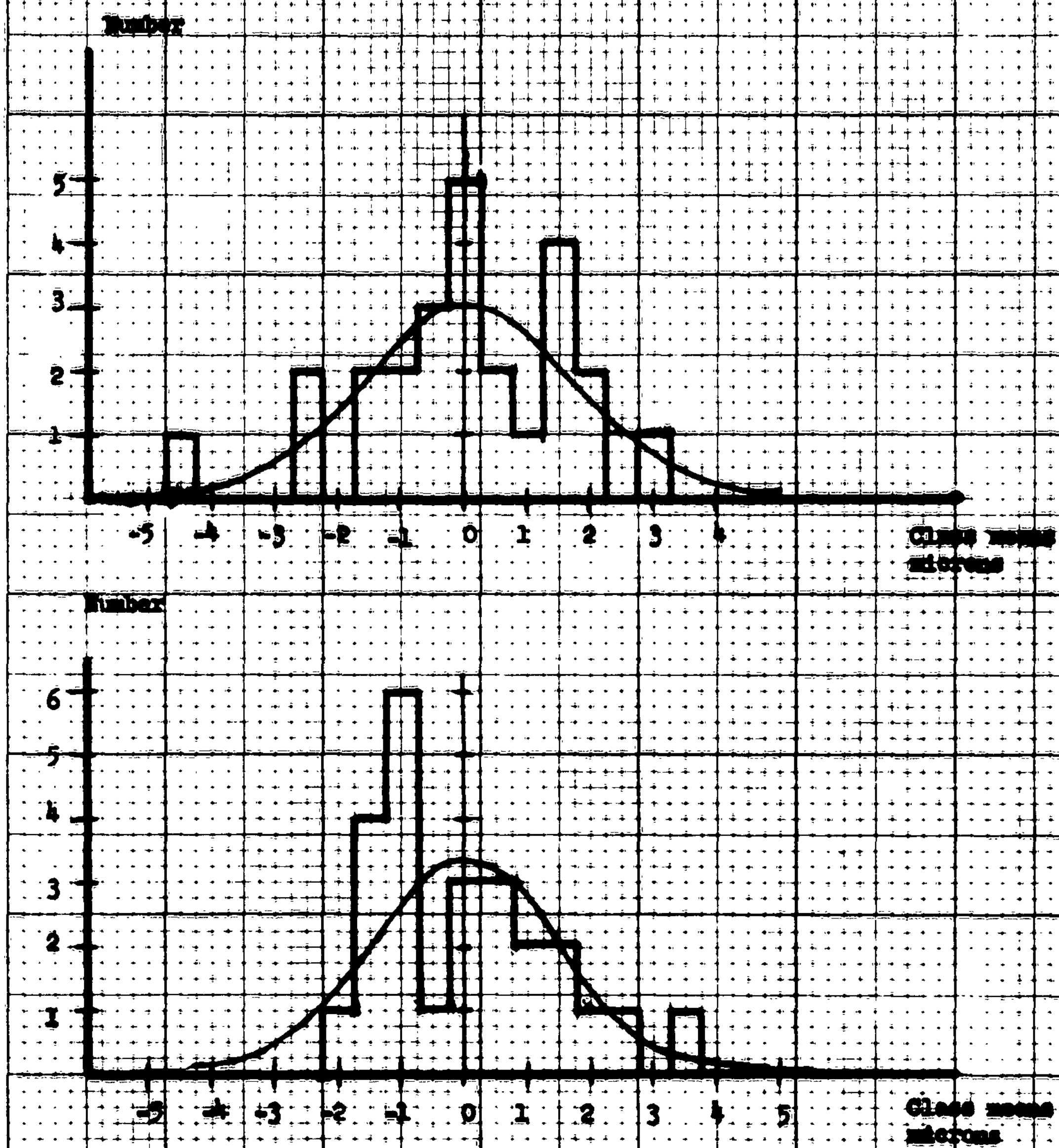
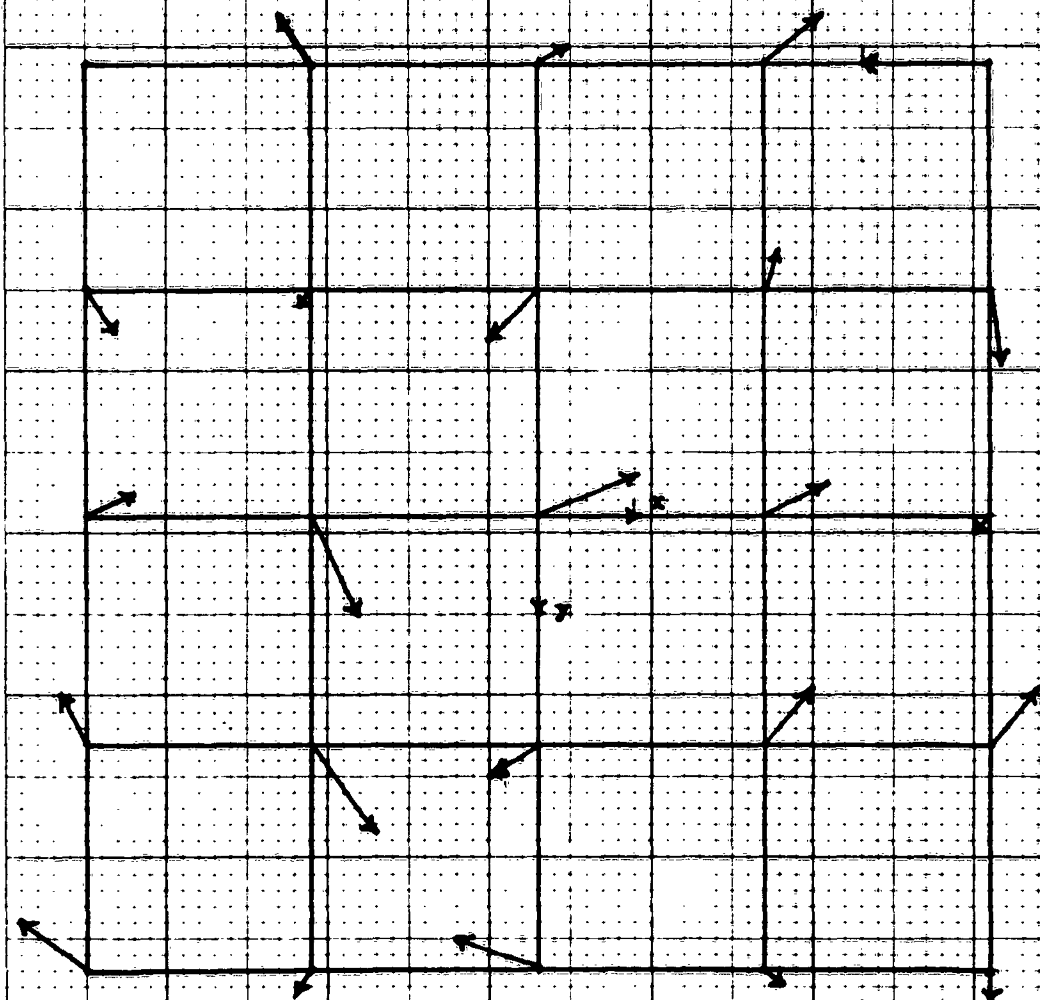


Figure 22. Histograms and normal distribution curves of the residuals in image coordinates after adjustment; x-coordinates top, y-coordinates bottom.



| | = 1 Micron

Figure 23. Graphical presentation of the residuals after the adjustment; all 25 points were used for the adjustment.

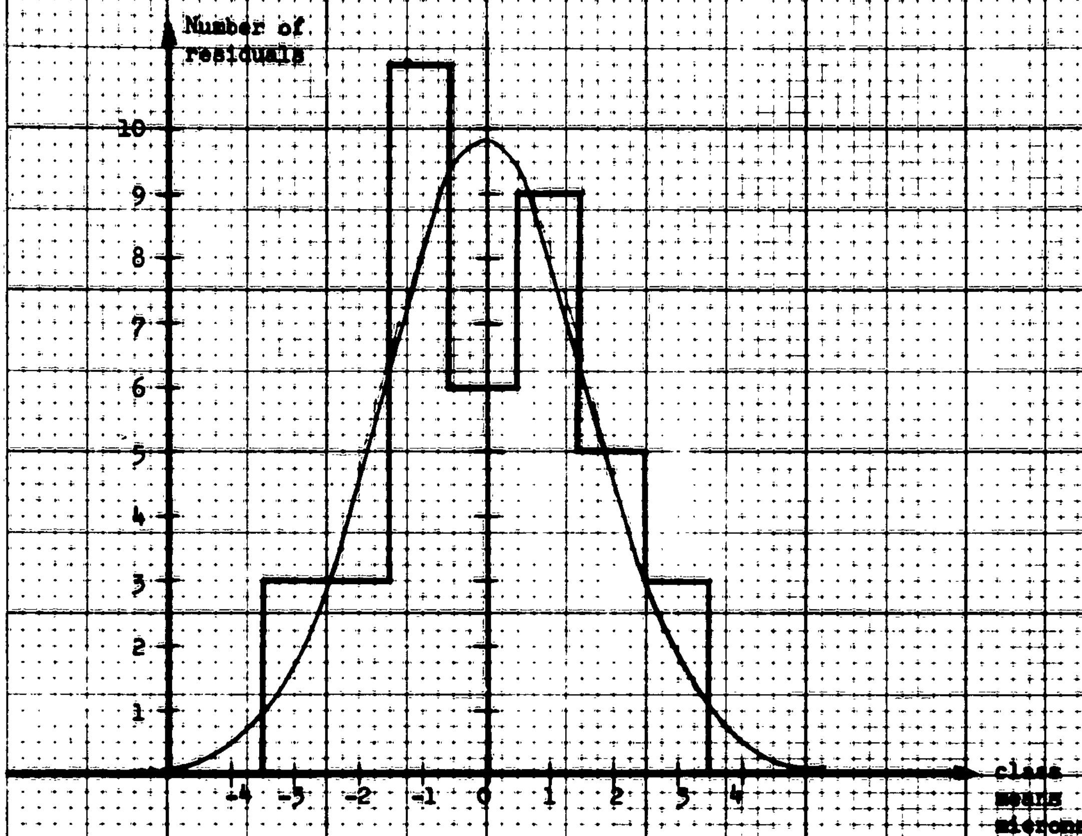


Figure 24. Histogram of the residuals in x and y after the adjustment in five points and the corresponding normal distribution curve.

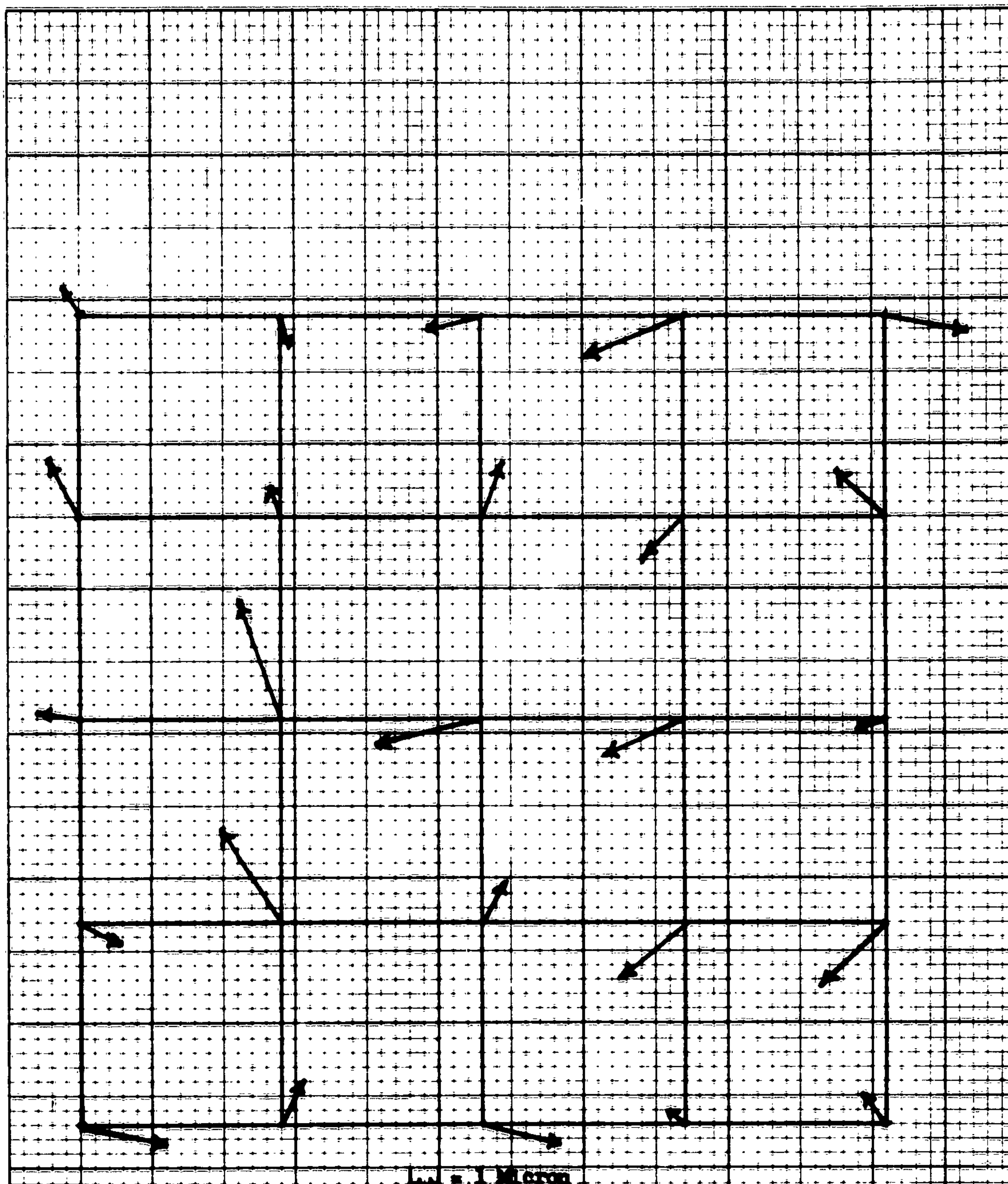


Figure 25. Graphical presentation of the residuals in the grid points after adjustment in the center point and the four corner points.

numerical corrections may be of less importance and, therefore, the actual final mechanical adjustment of the instruments may be neglected. The standard error of unit weight of image coordinate measurements in modern stereocomparators after adjustment has been found to be of the order of magnitude of 1 to 1.5 microns after correction for the standard error of the grid itself. The standard error of the coordinate measurements in connection with the scale determination of a comparator (measurements along a line) has been found to be of the order of magnitude of 0.5 microns, provided that at least five settings in each point are made and that the average of the readings is used.

From these basic data, confidence limits for all functions of the basic measurements can be established in order to obtain tolerances. For such a work, the theory of confidence limits from statistics has to be applied and a certain level has to be chosen. It should be emphasized, however, that more practical measurements in comparators, using grids of highest possible quality, are desired in order to obtain more information about the basic standard error of unit weight. Also for the routine tests of instrument and operators in connection with practical work, the derived methods and formula systems can be applied. Well-defined tolerances, founded upon the confidence limit theory, can serve as indications for acceptance or rejection.

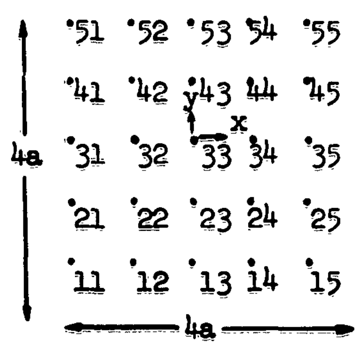
APPENDIX

Form 1

COMPUTATION FORMS

Comparator Tests. Determination of Coordinate Discrepancies.

Point Grid	x_{given} mm	$x_{meas.}$ mm	$x_{trans.}$ mm	dx $x_t - x_g$ microns	y_{given} mm	$y_{meas.}$ mm	$y_{trans.}$ mm	dy $y_t - y_g$ microns	Notes
11									
12									
13									
14									
15									
21									
22									
23									
24									
25									
31									
32									
33	000.000		000.000	0	000.000		0.000	0	
34									
35									
41									
42									
43									
44									
45									
51									
52									
53									
54									
55									



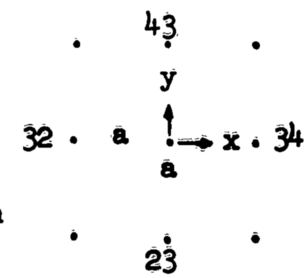
Positions and notations of grid points. Mann comparator system.

Comparator Tests

Form 2

Adjustment of Single Grid Measurements. 4 Points

Instrument: Date:
Operator:
Computed by: a = mm
Date:



$dx = x_{meas.} - x_{given}$; $dy = y_{meas.} - y_{given}$

Point	dx microns	k_1	k_1dx	k_2	k_2dx	k_3	k_3dx	dy microns	k_4	k_4dy	k_5	k_5dy	k_6	k_6dy
23				-1		0			+1				+2	
32		+1				+2					+1		+2	
43				+1		+2			-1					
34		-1				0					-1			
[dx]		N41		N43		[ ]		[dy]	N42		N44			
[dx^2]								[dy^2]						

$$dx_o = -\frac{[dx]}{4} =$$

$$dy_o = -\frac{[dy]}{4} =$$

$$dm_x = \frac{N41}{2a} =$$

$$dm_y = \frac{N42}{2a} =$$

$$d\alpha = \frac{N44}{2a} =$$

$$d\beta = \frac{N43 - N44}{2a} =$$

$$[vv] = [dx^2] + [dy^2] - \frac{[dx]^2 + [dy]^2}{4} - \frac{N41^2 + N42^2 + N43^2 + N44^2}{2} =$$

$$s_o = \sqrt{\frac{[VV]}{2}} =$$

Comparator Tests

Form 3

Adjustment of Single Grid Measurements. 9 Points

Instrument: Date:
Operator: a = mm
Computed by:
Date:

↑	51.	53.	55.
4a	31.	33. $\xrightarrow{x}$ (↑ y)	35.
↓	11.	13.	15.

$dx = x_{meas.} - x_{given}$; $dy = y_{meas.} - y_{given}$

Point	dx micr.	k_1	k_1dx	k_2	k_2dx	k_3	k_3dx	dy micr.	k_4	k_4dy	k_5	k_5dy	k_6	k_6dy
11		-1		+1		+1			-1		-1		-1	
13				+1		+2			-1				0	
15		+1		+1		+3			-1		+1		+1	
31		-1				0					-1		0	
33						+1							+1	
35		+1				+2					+1		+2	
51		-1		-1		-1			+1		-1		+1	
53				-1		0			+1				+2	
55		+1		-1		+1			+1		+1		+3	
[dx]=	↓	N91	↓	N94	↓		→	[dy] ↓	N92	↓	[ ]	↓		→
$[dx^2]$=								$[dy^2]$			N94			
											N93			

Corrections:

$dx_o = -\frac{[dx]}{9} =$ $s_{x_o} =$

$dy_o = -\frac{[dy]}{9} =$ $s_{y_o} =$

$dm_x = -\frac{N91}{12a} =$ $s_{m_x} =$

$dm_y = -\frac{N92}{12a} =$ $s_{m_y} =$

$d\alpha = \frac{N94-N93}{12a} =$ $s\alpha =$

$$d_\beta = \frac{N93-2N94}{12a} = \qquad s_\beta =$$

$$[vv] = [dx^2] + [dy^2] - \frac{[dx]^2 + [dy]^2}{9} - \frac{N91^2 + N92^2 + (N94-N93)^2 + N94^2}{6} =$$

$$=$$

$$s_o = \sqrt{\frac{[vv]}{12}} =$$

$$Q_{x_o x_o} = Q_{y_o y_o} = 0.1111; \quad Q_{m_x m_x} = Q_{m_y m_y} = Q_{\alpha\alpha} = \frac{1}{24a^2} =$$

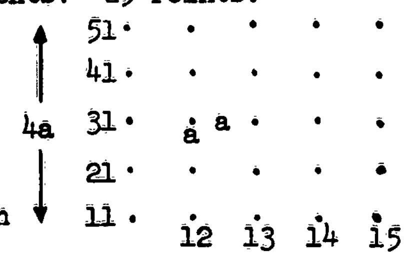
$$Q_{\beta\beta} = \frac{1}{12a^2} =$$

$$Q_{\alpha\beta} = -\frac{1}{24a^2} =$$

Adjustment of Single Grid Measurements. 25 Points.

Instrument: Date:
Operator: a = mm
Computed by: Date:

$dx = x_{meas.} - x_{given}$; $dy = y_{meas.} - y_{given}$



Point	dx micr.	k_1	k_1dx	k_2	k_2dx	k_3	k_3dx	dy micr.	k_4	k_4dy	k_5	k_5dy	k_6	k_6dy
11		+2		-2		+1			+2		+2		+5	
12		+1		-2		0			+2		+1		+4	
13				-2		-1			+2				+3	
14		-1		-2		-2			+2		-1		+2	
15		-2		-2		-3			+2		-2		+1	
21		+2		-1		+2			+1		+2		+4	
22		+1		-1		+1			+1		+1		+3	
23				-1		0			+1				+2	
24		-1		-1		-1			+1		-1		+1	
25		-2		-1		-2			+1		-2		0	
31		+2				+3					+2		+3	
32		+1				+2					+1		+2	
33						+1							+1	
34		-1				0					-1		0	
35		-2				-1					-2		-1	
41		+2		+1		+4			-1		+2		+2	
42		+1		+1		+3			-1		+1		+1	
43				+1		+2			-1				0	
44		-1		+1		+1			-1		-1		-1	
45		-2		+1		0			-1		-2		-2	
51		+2		+2		+5			-2		+2		+1	
52		+1		+2		+4			-2		+1		0	
53				+2		+3			-2				-1	
54		-1		+2		+2			-2		-1		-2	
55		-2		+2		+1			-2		-2		-3	
[dx] [dx²]		N251		N254		[ ]		[dy] [dy²]	N252		[ ] N254 N253		[ ]	

Corrections:

$$dx_o = -\frac{dx}{25} =$$

$$dy_o = -\frac{dy}{25} =$$

$$dm_x = \frac{N251}{50a} =$$

$$dm_y = \frac{N252}{50a} =$$

$$d\alpha = \frac{N253-N254}{50a} =$$

$$d\beta = \frac{2N254-N253}{50a} =$$

$$[vv] = [dx^2] + [dy^2] =$$

$$= \frac{[dx]^2 + [dy]^2}{25} =$$

$$= \frac{N251^2 + N252^2 + (N253-N254)^2 + N254^2}{50} =$$

$$=$$

$$s_o = \sqrt{\frac{[vv]}{44}} =$$

Comparator Tests

Adjustment of Four Sets of Measurements. 9 Points

Form for the Computation of the Factors T_1 - T_{10}

Form 5

Grid Position UI

Grid	dx_I	T_1		T_3		T_5		T_6, T_7		dy_I	T_2		T_4		T_7		S_y	
Point	micr.	k	kdx	k	kdx	k	kdx	k	kdx	micr.	k	kdy	k	kdy	k	kdy	k	kdy
11		-1		+1		+1		-1			-1		+1		+1		+2	
13						+1		-1			-1		+1				+1	
15		+1		-1		+1		-1			-1		+1		+1		0	
31		-1		+1											+1		+2	
33																	+1	
35		+1		-1											+1		0	
51		-1		+1		-1		+1			+1		-1		+1		+2	
53						-1		+1			+1		-1				+1	
55		+1		-1		-1		+1			+1		-1		-1		0	
$[dx]_I$										$[dy]_I$								

Grid Position UII

Grid	dx_{II}	T_1		T_4		T_5		T_8		dy_{II}	T_2		T_3		T_6, T_8		S_y	
Point	micr.	k	kdx	k	kdx	k	kdx	k	kdx	micr.	k	kdy	k	kdy	k	kdy	k	kdy
11		-1		+1		-1		+1			+1		-1		+1		+2	
13		-1		+1											+1		+2	
15		-1		+1		+1		-1			-1		+1		+1		+2	
31						-1		+1			+1		-1				+1	
33																	+1	
35						+1		-1			-1		+1				+1	
51		+1		-1		-1		+1			+1		-1		-1			
53		+1		-1											-1			
55		+1		-1		+1		-1			-1		+1		-1			
$[dx]_{II}$										[ ]								

Comparator Tests

Adjustment of Four Sets of Measurements. 9 points

Form for the Computation of the Factors $T_1 - T_{10}$

Form 6

Grid Position UIII

Grid Point	dx_{III} micr.	T_1 k	T_1 kdx	T_3 k	T_3 kdx	T_5 k	T_5 kdx	T_6,T_9 k	T_6,T_9 kdx	dy_{III} micr.	T_2 k	T_2 kdy	T_4 k	T_4 kdy	T_9 k	T_9 kdy	S_y k	S_y kdy
11		+1		-1		-1		+1			+1		-1		-1		0	
13						-1		+1			+1		-1				+1	
15		-1		+1		-1		+1			+1		-1		+1		+2	
31		+1		-1											-1		0	
33																	+1	
35		-1		+1											+1		+2	
51		+1		-1		+1		-1			-1		+1		-1		0	
53						+1		-1			-1		+1				+1	
55		-1		+1		+1		-1			-1		+1		+1		+2	
$[dx]_{III}$										[ ]								

Grid Position UIV

Grid Point	dx_{IV} micr.	T_1 k	T_1 kdx	T_4 k	T_4 kdx	T_5 k	T_5 kdx	T_{10} k	T_{10} kdx	dy_{IV} micr.	T_2 k	T_2 kdy	T_3 k	T_3 kdy	T_6,T_{10} k	T_6,T_{10} kdy	S_y k	S_y kdy
11		+1		-1		+1		-1			-1		+1		-1		0	
13		+1		-1											-1		0	
15		+1		-1		-1		+1			+1		-1		-1		0	
31						+1		-1			-1		+1				+1	
33																	+1	
35						-1		+1			+1		-1				+1	
51		-1		+1		+1		-1			-1		+1		+1		+2	
53		-1		+1											+1		+2	
55		-1		+1		-1		+1			+1		-1		+1		+2	
$[dx]_{IV}$										[ ]								

Summary of the Adjustment of Four Sets of Measurements. 9 Points.
U-position

Computations of the T-terms from forms 5 and 6:

$T_1 = T_{1xI} + T_{1xII} + T_{1xIII} + T_{1xIV} =$

$T_2 = T_{2yI} + T_{2yII} + T_{2yIII} + T_{2yIV} =$

$T_3 = T_{3xI} + T_{3yII} + T_{3xIII} + T_{3yIV} =$

$T_4 = T_{4yI} + T_{4xII} + T_{4yIII} + T_{4xIV} =$

$T_5 = T_{5xI} + T_{5xII} + T_{5xIII} + T_{5xIV} =$

$T_6 = T_{6xI} + T_{6yII} + T_{6xIII} + T_{6yIV} =$

$T_7 = T_{7xI} + T_{7yI} =$

$T_8 = T_{8xII} + T_{8yII} =$

$T_9 = T_{9xIII} + T_{9yIII} =$

$T_{10} = T_{10xIV} + T_{10yIV} =$

Computations of the Final Results:

$dm_{xc} - dm_{yc} = (T_2 - T_1):48a =$

$dm_{xg} - dm_{yg} = (T_4 - T_3):48a =$

$d\beta_c = -(2T_5 + T_7 + T_8 + T_9 + T_{10}):48a =$

$d\beta_g = (-2T_6 + T_7 + T_8 + T_9 + T_{10}):48a =$

$d\alpha_I = (-T_5 + T_6 - 3T_7 - T_8 - T_9 - T_{10}):48a =$

$d\alpha_{II} = (-T_5 + T_6 - 3T_8 - T_9 - T_{10}):48a =$

$d\alpha_{III} = (-T_5 + T_6 - T_7 - T_8 - 3T_9 - T_{10}):48a =$

$d\alpha_{IV} = (-T_5 + T_6 - T_7 - T_8 - T_9 - 3T_{10}):48a =$

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Research Note No. 3, 1 Aug 62, 95 pp, 25 illustrations, 23 tables
DA Task 8T35-12-001-02 Unclassified Report

Report covers development and practical application of test procedures to determine geometrical quality of comparators for image coordinate measurement. Procedures are founded upon grid coordinate measurements under operational conditions. First, basic principles for determination of accuracy of measurements have been treated under different assumptions concerning number and positions of test points. Principles of the method of least squares have been applied throughout, for determination of regular (systematic) errors of measured data as well as for estimation of a statistical value of irregular errors and for error propagation in functions of basic observations. Derivations have been made for grids, the given coordinates of which can be regarded to be errorless and for grids where certain regular errors are assumed to be present in the given coordinates. In the latter case no absolute scale can be determined and basic geometrical quality is in principle the precision. In a series of practical applications theoretical derivations have been used for testing a number of comparators of different types. Also some determinations of absolute scales have been performed. The lowest standard error of unit weight found in a comparator is of the order of magnitude 1 micron. Normal distribution tests of residuals have been performed throughout. From studies of residuals further regular errors may be detected and determined.

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Report covers development and practical application of test procedures to determine geometrical quality of comparators for image coordinate measurement. Procedures are founded upon grid coordinate measurements under operational conditions. First, basic principles for determination of accuracy of measurements have been treated under different assumptions concerning number and positions of test points. Principles of the method of least squares have been applied throughout, for determination of regular (systematic) errors of measured data as well as for estimation of a statistical value of irregular errors and for error propagation in functions of basic observations. Derivations have been made for grids, the given coordinates of which can be regarded to be errorless and for grids where certain regular errors are assumed to be present in the given coordinates. In the latter case no absolute scale can be determined and basic geometrical quality is in principle the precision. In a series of practical applications theoretical derivations have been used for testing a number of comparators of different types. Also some determinations of absolute scales have been performed. The lowest standard error of unit weight found in a comparator is of the order of magnitude 1 micron. Normal distribution tests of residuals have been performed throughout. From studies of residuals further regular errors may be detected and determined.

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U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency, Fort Belvoir, Virginia - DETERMINATION OF THE GEOMETRICAL QUALITY OF COMPARATORS FOR IMAGE COORDINATE MEASUREMENTS - K. Bertil P. Hallert

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